ALGEBRAIC DIVISION RING EXTENSIONS¹

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I. Introduction. A ring A is *radical over a subring* B in case some power $a^{n(a)}$ of each $a \in A$ lies in B . Some of the recent literature on the structure of rings has been devoted to the problem of finding general conditions on A which imply commutativity of A when B is the center of A and A/B is radical (see Kaplansky [10], Herstein [5], and especially Drazin [1] where much of the previous work along these lines is summarized and extended). This problem can be generalized in at least three ways. One can seek conditions on A which imply commutativity of A when A/B is radical and B is successively: (1) an arbitrary proper subring; (2) an arbitrary commutative subring; (3) a proper division (or even simple) subring.² The present paper includes a complete solution to the case (1). Cases (2) and (3) are fully treated in a following article [3]. In connection with (1) the main result is:

THEOREM A. *If a simple ring A with minimal one-sided ideals is radical over a proper subring B , then A is a (commutative) field.*

This is the best possible result in the sense that there exist noncommutative primitive rings with minimal one-sided ideals which are radical over proper (simple) subrings (see §IV). Whether or not Theorem A can be extended to arbitrary simple rings is an open problem (cf. Footnote 6 and Theorem 2). Note, however, until the existence of simple nil rings is ruled out, an affirmative solution to this problem will be beyond reach.

Theorem A has the following important implication for the generation of the noncommutative ring A : *If to each $a \in A$, $n(a)$ is a corresponding natural number, then A is generated by the set $\{a^{n(a)} \mid a \in A\}$.* This extends the result for division rings of Hua where $\{n(a) \mid a \in A\}$ is a bounded set and Kaplansky's generalization where $n(x^{-1}ax) = n(a)$ for all $x, a \in A$. Theorem A also has the following interesting corollary (extended in [3] to arbitrary primitive rings) which can be regarded as another natural generalization of the Wedderburn, Jacobson, and Kaplansky theorems for division rings: *To each non-central element a of a noncommutative simple ring A containing a*

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² I wish to take this opportunity to thank Professor M. Henriksen for suggesting (3), and for several stimulating conversations regarding Theorem A below.

minimal left ideal, there corresponds an element $b \in A$ such that no power of b commutes with a . Corollary 1[★], §IV, offers several more general results on the generation of simple algebras.

Results similar to Theorem A are obtained for extensions A/B of the Ikeda [7], Herstein [4], and Nakayama [11; 12] types. In fact, Theorem A is a by-product of results achieved in a far more general program described in the next paragraphs.

Let $S = S(\Phi)$ be a set of nonconstant polynomials with zero constant terms in the variable x over the field Φ . Such a collection of polynomials will be called an S -set. If A is an algebra over Φ , and if B is a subalgebra of A , then A/B is S -algebraic provided that to each $a \in A$ there exists $f_a(x) \in S$ such that $f_a(a) \in B$. Let $P(A, S, B)$ denote the proposition:

$P(A, S, B)$: If A is S -algebraic over B , then A is commutative.

An S -set is defined to be:

(1.1) an S^Δ -set if $P(A, S, B)$ is true whenever A is a division algebra, and B is the center of A ;

(1.2) an S^Δ -set if $P(A, S, B)$ is true whenever A is a division algebra, and B is a subalgebra $\neq A$;

(1.3) an $S^\star$ -set if $P(A, S, B)$ is true whenever A is a simple algebra with minimum condition, and B is a subalgebra $\neq A$.

In §§II and III (also see §V), every presently known S^Δ -set is shown to be an $S^\star$ -set. In §IV, the $S^\star$ -sets in turn, with certain restrictions on the coefficients of the polynomials (e.g., for every $f(x) \in S^\star$, $f(1) \neq 0$), are proved to be S^Δ -sets. For the latter S -sets, $P(A, S, B)$ is also true when A is a simple algebra with a minimal left ideal, and B is a subalgebra $\neq A$. All of which leaves the haunting question: Is every S^Δ -set an $S^\star$ -set?³

II. Radical division ring extensions. In this section the division ring case of Theorem A is established. D will always be a division ring, and Δ a subring of D . When D/Δ is radical, one easily verifies that Δ is necessarily a division subring (see Lemma 3, §III, in this connection), a fact which will be used hereafter without reference. Throughout this section Q' denotes the centralizer in D of any subset Q of D ; $d \in D$ is radical over Q if $d^n \in Q$ for some $n \geq 1$.

LEMMA 1. *Let D be a division ring and Δ a proper division subring.*

³ This can be answered affirmatively under the additional hypothesis that S^Δ forms a semi-group under composition of polynomials. (The proof of this, a modification of the proof of Theorem B below in which polynomials play the rôle of the powers X^n , will appear in a following article, *Radical extensions of rings*.) This immediately raises another question: Is the semi-group generated by an S^Δ -set also an S^Δ -set?

Assume that there exist elements $v, d \in D, d \in \Delta, v \notin \Delta$ such that d is not radical over v' . Then, at most one element in the set $\{(v+c)^{-1}d(v+c) \mid c \in d' \cap \Delta\}$ is radical over Δ .

PROOF. We actually show, for elements $v, d \in D, d \in \Delta, v \notin \Delta$, that if there exist different $c_i \in d' \cap \Delta$ such that $(v+c_i)^{-1}d(v+c_i), i=1$ and 2 , are both radical over Δ , then necessarily d is radical over v' . Accordingly, assume that there exist natural numbers k_i such that

$$[(v+c_i)^{-1}d(v+c_i)]^{k_i} \in \Delta,$$

where $d \in \Delta, v \notin \Delta, c_i \in d' \cap \Delta, i=1$ and 2 , and $c_1 \neq c_2$. Then,

$$(2.1) \quad d_i = (v+c_i)^{-1}d^n(v+c_i) \in \Delta,$$

$n = k_1 k_2, i=1$ and 2 , so that

$$(2.2+i) \quad v d_i + c_i d_i = d^n v + d^n c_i, \quad i = 1 \text{ and } 2.$$

By subtracting (2.4) from (2.3), and transposing the $c_i d_i$ to the right, one obtains

$$(2.5) \quad v(d_1 - d_2) = d^n(c_1 - c_2) + c_2 d_2 - c_1 d_1 \in -\Delta.$$

Now, $(d_1 - d_2) \neq 0$ would imply that $v \in \Delta$, contrary to its choice. On the other hand, setting $d_1 = d_2$ in (2.5) produces, since $(c_1 - c_2) \in (d^n)'$,

$$(2.6) \quad 0 = v0 = d^n(c_1 - c_2) - (c_1 - c_2)d_1 = (c_1 - c_2)(d^n - d_1).$$

Then, since $(c_1 - c_2) \neq 0$, (2.6) implies that $d^n = d_1$. If d_1 is replaced by d^n in (2.1), one sees that $(v+c_1) \in (d^n)'$. Since also $c_1 \in (d^n)'$, clearly $v \in (d^n)'$, as well. Evidently, $d^n \in v'$, contrary to the agreement that d is not radical over v' .

LEMMA 2. If D/Δ is radical, if $D \neq \Delta$, and if for every pair $v, d \in D$ satisfying $d \in \Delta, v \notin \Delta$, it follows that d is radical over v' , then D is a field.

PROOF. One first shows, for every pair $x, y \in D$, that y is radical over x' . If $x, y \in D, y \in \Delta, x \notin \Delta$, then y is radical over x' by hypothesis. Next assume both $x, y \in \Delta$, and choose $a \in D, a \notin \Delta$. Let n and m be such that $y^n \in a'$, and $(y^n)^m \in (x+a)'$. Then $y^k \in x', k = nm$, that is, y is radical over x' . The case $y \notin \Delta$ is reducible to the cases just decided since $y^n \in \Delta$ for suitable n . Next we show that D is commutative.⁴ Let $A = \Phi(x, y)$ denote the division subalgebra generated by the

⁴ Although this is an immediate consequence of the result just proved and Theorem 1 of Herstein's paper, *Two remarks on the commutativity of rings*, Canad. J. Math. vol. 7 (1955) pp. 411-412, the short proof given here makes no use of the axiom of choice (=Zorn's lemma) as required in the proof of Herstein's theorem.

pair $x, y \in D$, where Φ is the center of D . By the result just proved, if $a \in A$, then a^n commutes with both x and y for suitable n . Then A is radical over its center. By Kaplansky's theorem [10], A is commutative. Since x and y were arbitrary elements of D , D is commutative too.

We are now in a position to prove the division ring case of Theorem A, namely,

THEOREM B. *If D is a division ring which is radical over a proper subring Δ , then D is a field.*⁵

PROOF. The proof requires nothing more than a brief inspection of the results of Lemmas 1 and 2. If D is noncommutative, by Lemma 2, there exist elements $v, d \in D, v \notin \Delta, d \in \Delta$, such that d is not radical over v' . But then, Lemma 1 shows that some conjugate of d is not radical over Δ . This contradicts the hypothesis that D/Δ is radical. The only way out is to admit the commutativity of D .

Let D be a noncommutative division ring with center C . If $a \in D, a \notin C$, then, since D/a' is not radical, there exists $b \in D$ such that if σ is the inner automorphism effected by b , then different natural numbers n determine different conjugates $a\sigma^n, n=0, 1, 2, \dots$. This sharpens Herstein's theorem [*Conjugates in division rings*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 1021–1022] which states that a has infinitely many conjugates, and leads to the following result: If B is any noncentral subring, then there exists an inner automorphism σ of D such that different n determine different isomorphisms σ^n of B in D . It is an open question in the case B is at the same time a division subring $\neq D$ whether σ can be chosen so that the isomorphisms $B\sigma^n$ themselves are all different, $n=0, 1, 2, \dots$. {It is known that D always contains infinitely many distinct isomorphisms $x^{-1}Bx$ of B (cf. Theorem 3 of my article [*On conjugates in division rings*, Canad. J. Math. vol. 10 (1958) pp. 374–380]).}

III. On the generality of Theorem B. If D/Δ is a proper division ring extension, any condition which implies that D/Δ is radical implies commutativity of D . This principle underlies the proof of Theorem 1 below.

Let $[\alpha_1, \dots, \alpha_r, x]$ denote the subring of $\Phi[x]$ generated by x and r fixed nonzero elements $\alpha_1, \dots, \alpha_r$ in the field Φ , and set:

⁵ I wish to take this opportunity to thank Professor I. Kaplansky for reading a manuscript containing an earlier version of this theorem, and for words which encouraged. Moreover, the inspiration for this theorem was supplied by his beautiful theorem [10].

$$(3.1) \quad \begin{aligned} & N(\alpha_1, \dots, \alpha_r) \\ &= \{x^n - x^{n+1}p(x) \mid p(x) \in [\alpha_1, \dots, \alpha_r, x], n = 1, 2, \dots\} \end{aligned}$$

Polynomials of the type (3.1) have been studied in connection with commutativity of rings by Jacobson [8], Kaplansky [10], Ikeda [7], Nakayama [11], Herstein [4], and Nakayama again in a paper [12] in which the main results on commutativity of division rings in the papers just mentioned are extended.

The next lemma shows that the subalgebra Δ of Theorem 1 is really a division subalgebra if $\Delta \neq \{0\}$, a fact which is needed in the proof of Theorem 1.

LEMMA 3. *Let A be an algebra over a field Φ , with identity 1, and let B be a subalgebra of A such that A/B is S -algebraic. Then B contains b^{-1} together with every $b \in B$ which is regular in A .*

PROOF. Let $b \in B$ be regular in A , and let $g(x) = \sum_1^n \alpha_{n-i} x^i \in S$, $\alpha_0 \neq 0$, $\alpha_i \in \Phi$, $i = 0, \dots, n-1$, be such that $u = g(b^{-1}) \in B$. Then,

$$v = b^n u - \alpha_1 b - \dots - \alpha_{n-1} b^{n-1} = \alpha_0 1 \in B,$$

so that $\alpha_0^{-1}v = 1 \in B$. Furthermore,

$$w = b^{n-1}u - \alpha_1 1 - \dots - \alpha_{n-1} b^{n-2} = \alpha_0 b^{-1} \in B,$$

so that $\alpha_0^{-1}w = b^{-1} \in B$ as desired.

Below, $\Phi(d)$ is the division subalgebra generated by $d \in D$.

THEOREM 1. *Let D be a division algebra over the field Φ , and let Δ be a subalgebra such that to each $d \in D$ there corresponds nonzero elements $\alpha_1, \dots, \alpha_r \in \Phi$ (depending on d) such that for each $a \in \Phi(d)$ there exists $f_a(x) \in N_d$ satisfying $f_a(a) \in \Delta$, where $N_d = N(\alpha_1, \dots, \alpha_r)$ is a set of the type (3.1). Then D is a field.⁶*

PROOF. If $d \in D$, then $L = \Phi(d)$ is N_d -algebraic over $Z = L \cap \Delta$. If $\Delta = \{0\}$, then for each $d \in D$, $\Phi(d)$ is N_d -algebraic over the center of D . D is commutative in this case by Nakayama's [12, p. 23, Satz 1]. Now let $\Delta \neq \{0\}$. Then, Lemma 3 shows that Δ is a division subalgebra, hence $Z = L \cap \Delta$ is a field. If $d \notin \Delta$, then $L \neq Z$, so Nakayama's

⁶ Added in proof (November 10, 1959): A paper [A remark on the commutativity of algebraic rings, Nagoya Math. J. vol. 12 (1959) pp. 39-44] by T. Nakayama has appeared in which S -sets more general than those of (3.1) are shown to be S^Δ -sets. With references to pp. 42-43 of this paper (instead of [12]) the proof below verbatim shows that these are also S^Δ -sets. (Since these new S^Δ -sets are semigroups with respect to composition of polynomials, this fact can be deduced also from the result cited in Footnote 2.)

[12, p. 22, Hilfssatz 2] applies; L is either purely inseparable over Z , or else L is algebraic over a finite field. In either case, d is radical over Δ . Thus, $\Delta \neq \{0\}$ implies that D/Δ is radical. Then D is commutative by Theorem B.

Jacobson's [8, Theorem 8] states that every algebraic division algebra over a finite field is commutative. Expressed otherwise, every S -set contained in $\Phi[x]$ is an S^Δ -set if Φ is finite. The following corollary shows that every such S^Δ -set is an $S^\blacktriangle$ -set. This result is an immediate consequence of Theorem 1, since, if $\alpha_1, \dots, \alpha_r$ represent the nonzero elements of Φ , then clearly D/Δ is N -algebraic, where $N = N(\alpha_1, \dots, \alpha_r)$ is a set of the type (3.1).

COROLLARY. *Let D be a division algebra over a finite field Φ , and let Δ be a subalgebra $\neq D$ such that to each $a \in D$ there exists a nonconstant polynomial $f_a(x) \in \Phi[x]$ such that $f_a(a) \in \Delta$. Then D is a field.*

REMARK. Herstein's result of [6] can be applied together with Theorem B to give a proof which by-passes Theorem 1.

IV. Simple rings. In this section $A = T_n$ will always denote the complete algebra of $n \times n$ matrices over T , where T is an algebra with identity over the field Φ . As in the previous sections, D is a division algebra over Φ . If $A = D_n$, $n > 1$, is algebraic over Φ , and if Φ is finite, then it is known (cf., Rosenberg and Zelinsky [13]) that each $a \in A$ satisfies an equation of the form:

$$(4.0) \quad x^{n(a)} - x^{m(a)} = 0, \quad 0 < m(a) < n(a).$$

This is conclusive evidence that none of the theorems of §II can be extended *in toto* even to simple matrix algebras. Curiously enough, these results can be extended to simple algebras with minimal one-sided ideals when either of the following restrictions on the coefficients of the pertinent S -sets (violated in (4.0)⁷) is observed:

$$(4.1) \quad \text{If } f(x) = \sum_1^n \alpha_i x^i \in S, \text{ then } \alpha_1 \neq 0.$$

$$(4.2) \quad \text{If } f(x) \in S, \text{ then } f(1) = \sum_1^n \alpha_i \neq 0.$$

For convenience, any S -set satisfying (4.i) will be an S_i -set, $i = 1$ or 2 .

LEMMA 4. *Every $S_i^\blacktriangle$ -set is an $S_i^\star$ -set, $i = 1$ or 2 .*

PROOF. Let $A = D_n$, where D is a division algebra over Φ , and as-

⁷ The point being that the $m(a)$ of (4.0) must exceed 1 for some $a \in D_n$, when $n > 1$, and, clearly, $f(1) = 0$ whenever $f(x)$ has the form (4.0).

sume that A is S_i^Δ -algebraic over a subalgebra $B \neq A$, $i=1$ or 2 . If $n=1$, then $A=D$ is commutative in view of the definition (1.2). The method of proof is to show that $n=1$. Assume that $n>1$.

S_1^Δ CASE. Let u be any element of A such that $u^2=0$. If $f_u(u) \in S_1^\Delta$ has the form (4.1) and satisfies $f_u(u) \in B$, then $u = \alpha_1^{-1} f_u(u) \in B$. Hence, B contains all $e_{ij}d$, $d \in D$, $i \neq j$, where $\{e_{ij} \mid i, j=1, \dots, n\}$ are matrix units of A . Then, clearly, $B \supseteq A = D_n$, a contradiction.

S_2^Δ CASE. It is known (see [2, Proposition 1]) that A is generated as an algebra by idempotents. It is easy to see that B must contain all idempotents of A if A/B is S_2 -algebraic. Then $B=A$ is another contradiction.

By way of summary, all known $S^\star$ -sets are listed below:

$$(4.3) \quad N_1(\alpha_1, \dots, \alpha_r) = \{x - x^2 p(x) \mid p(x) \in [\alpha_1, \dots, \alpha_r, x]\},$$

$$(4.4) \quad N_2(\alpha_1, \dots, \alpha_r) = \{x^n - x^{n+1} p(x) \mid p(x) \in [\alpha_1, \dots, \alpha_r, x]; p(1) \neq 1; n = 1, 2, \dots\}.$$

For the sake of emphasis (see the corollary to Theorem 1), we also list

$$(4.5) \quad \text{Any } S_i\text{-set, } i=1 \text{ or } 2, \text{ contained in } \Phi[x], \text{ when } \Phi \text{ is finite.}$$

LEMMA 5. *Let A be an algebra over Φ , and let $\{B_\alpha \mid \alpha \in \Lambda\}$ be a non-empty collection of subalgebras such that each pair of elements of A is contained in some B_α . Suppose that each B_α is either commutative, or not S -algebraic over a proper subalgebra. Then A has the same property.*

PROOF. Assume A/B is S -algebraic over a subalgebra $B \neq A$. Choose $a, b \in A$, $b \in B$, $a \notin B$, and suppose $a, b \in B_\alpha$. Now B_α is a proper S -algebraic extension of $B \cap B_\alpha$. Since B_α is therefore commutative, every $b \in B$ commutes with every $a \in A$, $a \notin B$. But then it is easy to see that A is commutative.

PROPOSITION 1. *Let A be as in Lemma 5, only suppose if $R \in \{B_\alpha \mid \alpha \in \Lambda\}$ that $R = T_n$, $n = n(R)$, where $T = T(R)$ is either a division algebra over Φ , or else T is a commutative algebra over Φ . Then, if A is S_i^Δ -algebraic over a subalgebra $B \neq A$, $i=1$ or 2 , then A is commutative.*

PROOF. If $R = T_n$ is S_i^Δ -algebraic over a subalgebra $\neq R$, then $n=1$ by the arguments of Lemma 4. Then, $R=T$ is either *a priori* commutative, or commutative by the definition of an S^Δ -set. Then Lemma 5 applies.

Now let A be a simple algebra over Φ , and suppose that A has a minimal one-sided ideal. By Litoff's theorem (see [9, p. 90]) A is

locally a matrix algebra in the sense that any finite subset F of A is contained in a subalgebra of the form D_n , $n = n(F)$, where $D = D(F)$ is a division algebra over Φ . Thus A satisfies the hypotheses of Proposition 1. Since every commutative simple ring not a zero ring is a field, this establishes the next result which implies Theorem A.

THEOREM A[★]. *Let A be a simple algebra over a field Φ , and suppose that A contains a minimal one-sided ideal. Then, if A is S_i^Δ -algebraic over a subalgebra $B \neq A$, $i = 1$ or 2 , then A is a field.*

REMARK. In a similar fashion, one can prove that if the algebra A of Theorem A[★] is $S^\star$ -algebraic over a subalgebra $B \neq A$, then A is a field. This fact produces the following interesting results on the generation of simple algebras.

COROLLARY 1[★]. *Let A be a noncommutative simple (resp. division) algebra over Φ with a minimal left ideal, and let S be any $S^\star$ -set (resp. S^Δ -set). If a is any noncentral element of A , then there exists $b \in A$ such that $f(b)a \neq af(b)$ for all $f(x) \in S$. (2) To each $a \in A$ choose $f_a(x) \in S$. Then A is generated (as an algebra) by the set $\{f_a(a) \mid a \in A\}$.*

PROOF. (1) Since a is noncentral, its centralizer a' is a proper subalgebra of A . Since A is noncommutative, A cannot be S -algebraic over a' , that is, there exists $b \in A$ such that $f(b) \notin a'$ for all $f(x) \in S$. (2) Clearly A is S -algebraic over the subalgebra B generated by $\{f_a(a) \mid a \in A\}$. Since A is noncommutative, $A = B$.

An example given below shows that Theorem A cannot be extended to primitive rings, even if one assumes characteristic 0. Let L be the full ring of linear transformations (l.t.'s) in an infinite dimensional vector space M over an arbitrary division ring D , and let B (the socle) be the simple subring consisting of all $a \in L$ having finite dimensional range. Choose any $a \in L$, $a \notin B$, such that $a^n \in B$ for some n . For example, choose $a^2 = 0$ (I am indebted to Dr. Christine Williams Ayoub for the following remark: "Write $M = V \oplus W$, where V and W are isomorphic subspaces, and let a map V onto W , and W onto $\{0\}$ "). Clearly the subring A generated by B and a is radical over B . In fact, $b^2 \in B$ for any $b \in A$. Since A is a dense ring of l.t.'s, A is primitive (cf., e.g., [9, p. 175, Structure Theorem]). This shows that Theorem A cannot be extended, *in toto*, to primitive rings.⁸ Next suppose that D has characteristic 2, and let 1 be the identity

⁸ A partial result is contained in [3] where it is shown that if A is a ring with no nil ideals $\neq \{0\}$, and if A is radical over a subring B which is commutative with zero Jacobson radical (respectively, B a division subring $\neq A$), then A is commutative (respectively, A is a field). Cf. also the article cited in Footnote 3.

element of L . Then, if A_1 is the subring generated by B and 1 , $b^2 - b \in B$ for any $b \in A_1$. This shows that Theorem $A^\star$ cannot be extended to primitive rings either, when $S_1^\Delta = \{x^2 - x\}$. However, it is a fair question to ask whether Theorem A (and Theorem $A^\star$) can be extended to arbitrary simple algebras.⁶ If A is a simple algebra over a field $\Phi \neq GF(2)$, containing an idempotent $e \neq 0, 1$, it is known (cf., e.g., [2]) that A is generated (as an algebra) by: (1) the set of all $u \in A$ such that $u^2 = 0$; (2) the set of all idempotents in A . As indicated in the proof of Lemma 4, if A/B is S_i -algebraic, then B contains all elements in (i) , $i = 1$ or 2 . Thus, such an algebra A cannot be S_i -algebraic, $i = 1$ or 2 , over a subalgebra $B \neq A$, a fact which is used in the proof of the theorem below. In the statement, the term I -algebra is used in the sense of Jacobson [2, p. 210], namely, an algebra in which every non-nil left ideal contains a nonzero idempotent.⁹

THEOREM 2. *Let A be a simple I -algebra over a field $\Phi \neq GF(2)$, and assume that A is S_i^Δ -algebraic, $i = 1$ or 2 , over a subalgebra $B \neq A$. Then, either A is a field, or A is a nil algebra.*

PROOF. If J is the Jacobson radical of A , then either $J = A$, or $J = \{0\}$, by the simplicity of A . It is well-known that J contains no nonzero idempotents. Thus, either $J = A$ is nil, or $J = \{0\}$. Since J contains every nil left ideal of A , $J = \{0\}$ implies that every nonzero left ideal I of A contains a nonzero idempotent $e(I)$. Then A is a division algebra if and only if $e(I) = 1$ for every such I . In this case, A is commutative by the definition (1.2) of an S^Δ -set. The case $e(I) \neq 1$ for some nonzero left ideal I is ruled out by the remarks preceding Theorem 2.

It is an open question whether there exist simple nil algebras. In proving Theorem 2, two weaker (?) questions occurred to me: (1) Does there exist a simple algebra containing a nonzero nil left ideal?¹⁰ (2) Does there exist a simple algebra containing a nonzero nilpotent element all of whose algebraic elements are nilpotent?

Theorem 2 offers a criterion for deciding whether a noncommutative simple algebraic algebra (or simple I -algebra) A over a field $\Phi \neq GF(2)$ is nil: A is nil if and only if A is radical over a subalgebra $B \neq A$. Then A is radical over every subalgebra!

⁹ Every algebraic algebra is an I -algebra [2, p. 210, Proposition 1]. An I -algebra is sometimes called a Zorn algebra.

¹⁰ An affirmative answer would have an important application: Then, either A is a simple nil ring, or else the Köthe radical of A does not exist.

V. Conclusion. Theorem 1 can be generalized to the case D is a division algebra (in the sense of Jacobson [9, p. 16]) over a commutative integral domain ϕ (with identity), and Δ is a ϕ -subalgebra, by requiring for each $d \in D$ that:

(1) the pertinent polynomials $f_a(x)$ in the set N_d (defined by (3.1) in the polynomial ring $\phi[x]$ over ϕ) are all monic.

(2) $f_a(a) \in \Delta$, $f_a(x) \in N_d$, for each $a \in \Phi(d)$, where Φ is the field of quotients of ϕ , where 1 is the identity of D .

The value of (1) is that the proof of Lemma 3 can be carried over verbatim to show that Δ is a Φ -subalgebra of D considered as an algebra over Φ . (In fact, when $\Delta \neq \{0\}$, Δ is actually a division subring containing Φ .) This reduces the situation to precisely that of Theorem 1.

THEOREM. *If A is a simple algebra with a minimal left ideal over a commutative integral domain ϕ , and if there exist a subalgebra $B \neq A$ and nonzero elements $\alpha_1, \dots, \alpha_r \in \phi$ such that to each $a \in A$ there corresponds a monic polynomial $p_a(x)$ in the subring of $\phi[x]$ generated by $\alpha_1, \dots, \alpha_r$, and x such that $a^N - a^{N+1}p_a(a) \in B$ for a natural number $N = N(a)$ subject to the requirement that $N(a) = 1$ for all $a \in A$ if $p_a(1) \neq 0$ for some $a \in A$, then A is a field.*

PROOF. If $p_a(I) = 0$ for all (idempotent) a , then B contains all idempotents; if $N(a) = 1$ for all (nilpotent) a , B contains all nilpotent elements of index 2. In either case, by the proof of Lemma 4, A is then a division ring satisfying (1) and (2) above.

Added in proof (November 10, 1959). For a long time I thought that every S^Δ -algebraic field extension was also a radical extension. I wish to present a counter example, and, at the same time, a new class of S^Δ -sets, which hinge on the notion of quasi-algebraically closed (q.a.c.) fields. Here, a field ϕ is q.a.c. in case every finite-dimensional division algebra over ϕ is commutative. The q.a.c. fields include: (1) algebraically closed fields; (2) Galois fields; and (3) any field of rational functions (in a single variable) with coefficients in a field of the type (1). C. C. Tsen has proved that any field ϕ of type (3) is q.a.c. but the field of rational functions over (3) is not q.a.c. [*Algebren über Funktionenkörpern*, Göttingen dissertation, 1934]. (I am indebted to F. Kasch for this reference to Tsen's theorem.)

Clearly, any locally finite division algebra over a q.a.c. field is commutative; for example, if D is an algebraic algebra over a q.a.c. field, and if D is finite-dimensional over its center C , then $D = C$, since D is evidently locally finite (cf., e.g., [9, p. 242, Proposition 3, or Theorem 1]). This leads to a new class of S^Δ -sets.

THEOREM. *Let ϕ be a q.a.c. field; let n be a fixed natural number; and let S_n be any $S(\phi)$ -set whose elements all have degrees $\leq n$. Then S_n is an S^Δ -set.*

PROOF. Let D be an algebra over ϕ such that D/C is S_n -algebraic. Since D/C is thereby algebraic of bounded degree, $[D:C] < \infty$ by [8, Theorem 7]. If $D \neq C$, there exists $a \in D$, $a \notin C$ such that the field $L = C(a)$ is separable over C (cf., e.g. [8, Lemma 2]). Since L/C is $S(\phi)$ -algebraic, and $L \neq C$, by Herstein's theorem [6], L , whence D , is algebraic over ϕ . Since D/ϕ is locally finite, as indicated above, $D = C$. Thus, S_n is an S^Δ -set.

Now let ϕ be a field (3). If k is a finite algebraic field extension of ϕ of degree $n > 1$, then to each $a \in K$ there corresponds a polynomial $f_a(x)$ in the polynomial ring over ϕ such that: (i) $f_a(a) \in \phi$; (ii) degree $f_a(x) \leq n$; and (iii) $S_n = \{f_a(x) \mid a \in k\}$ is an $S(\phi)$ -set. By the theorem, S_n is an S^Δ -set. If the characteristic of ϕ is properly chosen, say, characteristic $= 0$, then one can be sure by Kaplansky's lemma [10], that k/ϕ is not radical. Then, k/ϕ is an S^Δ -algebraic field extension which is not a radical extension. (It can be shown [see the paper cited in Footnote 4] that $S_n = S^\Delta$ is also an S^Δ -set.)

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