A MARTINGALE INEQUALITY AND THE LAW OF LARGE NUMBERS[1]

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In a recent paper [4], Hájek and Rényi have generalized an inequality of Kolmogorov to the following: If $x_1, x_2, \cdots, x_m$ are independent random variables with

$$E(x_k) = 0$$

and

$$E(x_k^2) < \infty, \qquad (k = 1, 2, \cdots, m)$$

and $c_1 \geqq c_2 \geqq \cdots > 0$, for any $\epsilon > 0$

$$P\left\{\max_{m \geq k \geq 1} c_k \left| x_1 + \cdots + x_k \right| \geqq \epsilon\right\} \leqq \frac{1}{\epsilon^2} \sum_1^m c_k^2 E(x_k^2). \tag{1}$$

The original Kolmogorov's inequality [6] has been extended to a martingale inequality by Lévy [8] and Ville [12] and later to a semi-martingale inequality by Doob [3]. In this note we will extend (1) to a semi-martingale inequality which contains Doob's inequality as a special case. As Kolmogorov's inequality is the key to the proof of the law of large numbers for a sequence of independent random variables, we will use our inequality to prove a "law of large numbers" for a martingale, which will be shown to include the extensions of Kolmogorov's law of large numbers for independent random variables [7] made by Brunk [1], Chung [2], Kawata and Udagawa [5], and Prohorov [11], and for dependent random variables made by Lévy [8] and Loève [9].

In the following (W, F, P) will be a probability space, $c_1, c_2, \cdots$ a nonincreasing sequence of positive numbers, $x_1, x_2, \cdots$ a sequence of random variables, $y_k = x_1 + x_2 + \cdots + x_k$ and F_k the Borel field generated by $x_1, x_2, \cdots, x_k$ for each k, and for a random variable z we put $z^+ = \max(z, 0)$.

THEOREM 1. *Let* (y_k) *be a semi-martingale relative to* (F_k) [3, *p.* 294] *and* $\epsilon > 0$. *Then*

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\begin{aligned}
\epsilon P\left\{\max_{m\ge k\ge 1} c_k y_k \geqq \epsilon\right\} &\leqq c_1 E(\overset{+}{y}_1) + \sum_2^m c_k E(\overset{+}{y}_k - \overset{+}{y}_{k-1}) \\
&\qquad - c_m \int_{\left\{\max_{m\ge k\ge 1} c_k y_k < \epsilon\right\}} \overset{+}{y}_m dP \\
&\leqq c_1 E(\overset{+}{y}_1) + \sum_2^m c_k E(\overset{+}{y}_k - \overset{+}{y}_{k-1}) \\
&= \sum_1^{m-1} (c_k - c_{k+1}) E(\overset{+}{y}_k) + c_m E(\overset{+}{y}_m).
\end{aligned} \tag{2}
$$

To prove (2), let

$$
A = \left\{\max_{m\ge k\ge 1} c_k y_k \geqq \epsilon\right\},
$$

$A_k = \{c_j y_j < \epsilon$ for $1 \leqq j < k$; $c_k y_k \geqq \epsilon\}$, and $z_k = y_k^+$ for $1 \leqq k \leqq m$. Then $A = \bigcup_1^m A_k$ and $A_k \in F_k$ for each k. Hence

$$
\begin{aligned}
\epsilon P(A) &= \epsilon \sum_1^m P(A_k) \leqq \sum_1^m c_k \int_{A_k} y_k dP = \sum_1^m c_k \int_{A_k} z_k dP \\
&= c_1 E(z_1) - c_1 \int_{W-A_1} z_1 dP + \sum_2^m c_k \int_{A_k} z_k dP = c_1 E(z_1) \\
&\quad + c_2 \int_{W-A_1} (z_2 - z_1) dP - c_2 \int_{W-(A_1\cup A_2)} z_2 dP + \sum_3^m c_k \int_{A_k} z_k dP.
\end{aligned}
$$

By the semi-martingale property, $\int_{A_1\cup A_2\cup\cdots\cup A_k}(z_{k+1} - z_k)dP \geqq 0$ for each k, then

$$
\begin{aligned}
\epsilon P(A) &\leqq c_1 E(z_1) + c_2 E(z_2 - z_1) - c_2 \int_{W-(A_1\cup A_2)} z_2 dP + \sum_3^m c_k \int_{A_k} z_k dP \\
&\leqq c_1 E(z_1) + c_2 E(z_2 - z_1) + c_3 \int_{W-(A_1\cup A_2)} (z_3 - z_2) dP \\
&\quad - c_3 \int_{W-(A_1\cup A_2\cup A_3)} z_3 dP + \sum_4^m c_k \int_{A_k} z_k dP \\
&\leqq \cdots \leqq c_1 E(z_1) + \sum_2^m c_k E(z_k - z_{k-1}) - c_m \int_{W-A} z_m dP \\
&\leqq c_1 E(z_1) + \sum_2^m c_k E(z_k - z_{k-1}).
\end{aligned}
$$

Thus the proof is complete.

If $x_1, x_2, \cdots$ are independent with mean zero and finite variance, then (y_k^2) is a semi-martingale relative to (F_k) [**3**, p. 294] and $E(y_k^2) = E(x_1^2 + \cdots + x_k^2)$, and therefore (2) reduces to (1). If $c_k = 1$ for each k, then $c_k - c_{k+1} = 0$ and (2) reduces to Doob's inequality [**3**, p. 314].

As an application of Theorem 1 we have:

COROLLARY. *Let (y_k) be a non-negative semi-martingale relative to (F_k) and* $\lim c_k = 0$. *If for some $\alpha \geqq 1$ $E(y_k^\alpha) < \infty$ for each k and*

$$\sum_2^\infty c_k^\alpha E(y_k^\alpha - y_{k-1}^\alpha) < \infty, \tag{3}$$

then

$$\lim c_k y_k = 0 \ a.e. \tag{4}$$

Since (y_k^α) is semi-martingale [**3**, p. 295], by Theorem 1 for $\epsilon > 0$, we have

$$\begin{aligned} \epsilon^\alpha P\left\{\sup_{k \geq n} c_k y_k \geqq \epsilon\right\} &= \epsilon^\alpha P\left\{\sup_{k \geq n} c_k^\alpha y_k^\alpha \geqq \epsilon^\alpha\right\} \\ &\leqq c_n^\alpha E(y_n^\alpha) + \sum_{n+1}^\infty c_k^\alpha E(y_k^\alpha - y_{k-1}^\alpha). \end{aligned}$$

By Kronecker's lemma [**9**, p. 238] and (3),

$$\lim c_n^\alpha E(y_n^\alpha) = 0,$$

and then

$$\lim P\left\{\sup_{k \geq n} c_k y_k \geqq \epsilon\right\} = 0.$$

Hence (4) holds under the condition (3).

When (y_k) is a martingale and $E(y_k^2) < \infty$ for each k, (y_k^2) is a semi-martingale [**3**, p. 295] and $E(y_k^2) = E(x_1^2) + \cdots + E(x_k^2)$ [**3**, p. 92]. Therefore, if x_k's are uniformly bounded and $c_k = 1/k$ the corollary gives Lévy's result [**8**, p. 252], and if $\alpha = 2$ the corollary reduces to Loève's extension [**9**, p. 387] of Lévy's result.

In the following, A will denote a constant, not necessarily always the same, depending on α and β.

THEOREM 2. *Let (y_k) be a martingale relative to (F_k),* $\lim c_k = 0$, $\alpha \geqq 1$ *and* $2\alpha \geqq \beta > 0$. *If for* $i \geqq i_0$

$$E(|y_i|^{2\alpha}) \leqq AE\left(\sum_1^i x_k^2\right)^\alpha, \tag{5}$$

$$i^{\alpha-1} c_i^{2\alpha-\beta} \leqq A, \qquad \sum_i^\infty c_k^{2\alpha} k^{\alpha-2} \leqq A c_i^\beta, \tag{6}$$

and

$$\sum_1^\infty c_k^\beta E(|x_k|^{2\alpha}) < \infty, \tag{7}$$

then (4) *is true.*

In the proof we may assume that $i_0 = 1$. By the Hölder's inequality,

$$E(|y_k|^{2\alpha}) \leqq Ak^{\alpha-1} \sum_1^k E(|x_i|^{2\alpha}). \tag{8}$$

By (8), (6) and Kronecker's lemma,

$$\begin{aligned} \lim c_k^{2\alpha} E(|y_k|^{2\alpha}) &\leqq A \lim c_k^{2\alpha} k^{\alpha-1} \sum_1^k E(|x_i|^{2\alpha}) \\ &\leqq A \lim c_k^\beta \sum_1^k E(|x_i|^{2\alpha}) = 0. \end{aligned}$$

Again by (8),

$$\begin{aligned} \sum_1^\infty (c_k^{2\alpha} - c_{k+1}^{2\alpha}) E(|y_k|^{2\alpha}) &\leqq A \sum_1^\infty (c_k^{2\alpha} - c_{k+1}^{2\alpha}) k^{\alpha-1} \sum_1^k E(|x_i|^{2\alpha}) \\ &= A \sum_1^\infty E(|x_i|^{2\alpha}) \sum_i^\infty (c_k^{2\alpha} - c_{k+1}^{2\alpha}) k^{\alpha-1}. \end{aligned}$$

Now by (6)

$$\begin{aligned} \sum_i^\infty (c_k^{2\alpha} - c_{k+1}^{2\alpha}) k^{\alpha-1} &= c_i^{2\alpha} i^{\alpha-1} + \sum_{i+1}^\infty c_k^{2\alpha}((k+1)^{\alpha-1} - k^{\alpha-1}) \\ &\leqq c_i^{2\alpha} i^{\alpha-1} + A \sum_{i+1}^\infty c_k^{2\alpha} k^{\alpha-2} \leqq A c_i^\beta. \end{aligned}$$

Hence

$$\sum_1^\infty (c_k^{2\alpha} - c_{k+1}^{2\alpha}) E(|y_k|^{2\alpha}) \leqq A \sum_1^\infty c_i^\beta E(|x_i|^{2\alpha}) < \infty,$$

and (4) is true by the corollary.

If $x_1, x_2, \cdots$ are independent with mean zero, then (5) is satisfied by an inequality due to Marcinkiewicz and Zygmund [10, Theorem

13]. If there is a subsequence c_{n_k} of c_n such that $1 < r \leqq c_{n_k}/c_{n_{k+1}} \leqq r' < \infty$ and $c_k \leqq A/k$, then (6) is satisfied with $\beta = \alpha + 1$, since for $n_{k_0} \leqq i < n_{k_0}$ we have

$$i^{\alpha-1} c_i^{2\alpha-\beta} \leqq A$$

and

$$\begin{aligned}\sum_{k=i}^{\infty} c_k^{2\alpha} k^{\alpha-2} &\leqq \sum_{n_{k_0}}^{\infty} c_k^{2\alpha} k^{\alpha-2} \leqq \sum_{j=k_0}^{\infty} c_{n_j}^{2\alpha} \sum_{k=n_j}^{n_{j+1}-1} k^{\alpha-2} \\ &\leqq A \sum_{k_0}^{\infty} c_{n_j}^{2\alpha} n_{j+1}^{\alpha-1} \leqq A \sum_{k_0}^{\infty} c_{n_j}^{\alpha+1} n_{n_{j+1}}^{\alpha-1} n_{j+1}^{\alpha-1} \\ &\leqq A \sum_{k_0}^{\infty} c_{n_j}^{\alpha+1} \leqq A c_{n_{k_0}}^{\alpha+1} \left(1 + \frac{1}{r} + \frac{1}{r^2} + \cdots \right) \leqq A c_{n_{k_0}}^{\alpha+1} \\ &\leqq A c_{n_{k_0}+1}^{\alpha+1} \leqq A c_i^{\alpha+1}.\end{aligned}$$

Therefore Theorem 2 includes the results obtained by Brunk, Chung, Kawata and Udagawa, and Prohorov. It is easy to verify that (6) is satisfied by $c_k = k^{-r}$ for $r > 0$ and $\beta = 2\alpha - (\alpha - 1)/r > 0$, and by $c_k = k^{-k}$ where β is any positive number less than 2α. The last case, $c_k = k^{-k}$, gives an example that the usual condition $\limsup c_k/c_{k+1} < \infty$ for $\lim c_k y_k = 0$ a.e. for the independent random variable case is not necessary.

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