A SUFFICIENT CONDITION FOR A MATRIC FUNCTION TO BE A PRIMARY MATRIC FUNCTION[1]

WALTER O. PORTMANN[2]

1. **Introduction.** A primary matric function is defined to be a matric function (that is, a mapping whose range and domain are sets of $n \times n$ matrices) arising from a scalar function of a complex variable. It has been shown [1] that primary matric functions are H-analytic. In this paper other necessary conditions for a primary matric function will be exhibited and it will then be shown that these conditions are also sufficient for a matric function to be a primary function.

We will first use a form of the definition of a primary function proposed by Frobenius and later use an equivalent form proposed by Giorgi [4]. Frobenius proposed that if the scalar function $f(z)$ is analytic at the eigenvalues of Z in $\mathfrak{M}$ (the algebra of square matrices of order n over the complex field) then $f(Z)$ shall be defined by

$$f(Z) = \frac{1}{2\pi i}\int_C \frac{f(\lambda)}{\lambda I - Z}\, d\lambda, \tag{1.1}$$

where C is a set of admissible closed paths enclosing each of the distinct eigenvalues of Z. That is, the components of $f(Z)$ are the integrals over C of the corresponding components of the matrix $f(\lambda)(\lambda I - Z)^{-1}/2\pi i$.

We wish to exhibit sufficient conditions on a matric function $F(Z)$ such that there will exist a scalar function $g(z)$ for which $F(Z) = g(Z)$ where $g(Z)$ may be computed as in (1.1).

2. **Necessary conditions.** It has previously been shown in [1] that primary matric functions are H-analytic in $\mathfrak{M}$, that is, the component functions of a primary function $g(Z)$ are analytic functions of the components z_{ij} of Z, for Z in an $\mathfrak{M}$-neighborhood of a matrix at which $g(Z)$ is defined.

If $g(z)$ is a scalar function defined at a matrix X, that is, $g(z)$ is analytic at the eigenvalues of X, and if Y is such that for some nonsingular matrix P, $Y = P^{-1}XP$, then g is defined at Y and $g(Y) = P^{-1}g(X)P$, as can be seen from (1.1).

If Z is a matrix whose eigenvalues lie in the domain of analyticity of $g(z)$, then the r, s component of $g(Z)$ is given by

Received by the editors January 3, 1959 and, in revised form, April 13, 1959.

[1] This paper was prepared under the facilities granted by the Case Research Fund.

[2] The author is now associated with Arizona State University.

$$g(Z)_{rs} = \frac{1}{2\pi i}\int_C g(\lambda)(\lambda I - Z)_{rs}^{-1}\,d\lambda,$$

where $(\lambda I-Z)_{rs}^{-1}$ is the r, s component of $(\lambda I-Z)^{-1}$. For an upper triangular matrix $Z=(z_{ij})$, $z_{ij}=0$ for $i>j$, a simple computation shows that $(\lambda I-Z)_{rs}^{-1}$ and thus $g(Z)_{rs}$ depend only on the z_{ij} for which $r\leqq i\leqq j\leqq s$ and is zero for $r>s$. In particular, $g(Z)_{rr}=g(r_{rr})$ for Z a diagonal (or upper triangular) matrix.

3. Sufficient conditions. We shall now show that these necessary conditions are also sufficient. For convenience the norm of a matrix $Z=(z_{ij})$ shall be defined by $\operatorname{norm}(Z)=\max_{i,j}|z_{ij}|$.

THEOREM 3.1. *Let D be an open domain of H-analyticity of a matric function F on $\mathfrak{M}$.*

(i) *Let F be such that X in D and $Y=P^{-1}XP$ implies that Y is in D and $F(Y)=P^{-1}F(X)P$.*

(ii) *Let F also be such that if $T=(t_{ij})$, in D, is a diagonal matrix, then $F(T)_{rr}$ is a function of only t_{rr}, where $F(T)_{rr}$ is the r, r component of $F(T)$, that is*

$$F(T)_{rr} = g_{rr}(t_{rr}).$$

Then there exists a scalar function $g(z)$ such that for all Z in D, $g(Z)=F(Z)$.

PROOF. Let C be a Jordan form for a matrix Z at which F is H-analytic, then C is a direct sum $C_{p_1}\dotplus\cdots\dotplus C_{p_k}$ of canonical blocks of the form

$$C_{p_i} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ & \lambda_i & 1 & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \lambda_i & 1 \\ 0 & & & & \lambda_i \end{pmatrix}$$

with p_i rows and columns. (The λ_i occurring in different C_{p_i} need not be distinct.)

From (i) and Lemma 4.1 of [2] it follows that $F(C)$ commutes with all matrices that commute with the canonical matrix C. It is known that a matrix $F(C)$ satisfying this condition must be a direct sum $P_1(C_{p_1})\dotplus\cdots\dotplus P_k(C_{p_k})$, where

$$(3.1)\qquad P_i(C_{p_i}) = \begin{pmatrix} \alpha_{i_1} & \alpha_{i_2} & \alpha_{i_3} & \cdots & \alpha_{i_{p_i}} \\ & \alpha_{i_1} & \alpha_{i_2} & \cdots & \alpha_{i_{p_i}-1} \\ & & \alpha_{i_1} & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & \ddots & \vdots \\ 0 & & & \alpha_{i_1} & \alpha_{i_2} \\ & & & & \alpha_{i_1} \end{pmatrix}$$

and $\alpha_{i_m}=\alpha_{j_m}$ for $\lambda_i=\lambda_j$ (see Turnbull and Aitken [7]).

Now, using a definition proposed by G. Giorgi which is equivalent to (1.1) [4] for $g(Z)$ where $g(z)$ is a scalar function, it is seen that the theorem will be proven if there exists a scalar function $g(z)$ such that, for $C=P^{-1}ZP$, where Z is any matrix at which F is H-analytic,

$$(3.2)\qquad \alpha_{j_m} = g^{(m-1)}(\lambda_j)/(m-1)!$$

or,

$$(3.3)\qquad F(C)_{r_jr_j+i} = g^{(i)}(\lambda_j)/i!, \qquad j = 1, \cdots, k,\ i = 0, \cdots, p_j - 1,$$

where $F(C)_{r_jr_j+i}$ is the r_j, r_j+i component of $F(C)$ (that is, $F_{r_jr_j+i}$ evaluated at the components of C) and $r_j=1+\sum_{t=1}^{j-1} p_t$ for $1<j\leqq k$, $r_j=1$ for $j=1$. (This choice of r_j proves (3.2) for components in the first row of each triangular block $P_j(C_{p_j})$ of $F(C)$ associated with C_{p_j} of C, which is all that is necessary, since in any such block, the values on any super diagonal are all equal.)

We shall first exhibit a scalar function $g(z)$ which is determined by F and then show that this function has the required property (3.3).

Let Z be an arbitrary but fixed matrix such that F is H-analytic in a neighborhood of Z; then Z is similar to an upper triangular matrix $X=(x_{ij})$ whose eigenvalues are the x_{ii}. By (i) F is H-analytic in a neighborhood of X. Choose any matrix $Y=(y_{ij})$ such that $y_{ij}=x_{ij}$ and $y_{ii}\neq y_{jj}$ for $i\neq j$, and $|y_{ii}-x_{ii}|<\epsilon$, where ϵ is sufficiently small such that F is H-analytic at Y (such an ϵ exists since F is H-analytic in a neighborhood of X). Y is similar to a diagonal matrix $A=\operatorname{diag}(y_{kk})$ with distinct eigenvalues y_{ii} and by (i) F is H-analytic at A. Now, A is similar to a diagonal matrix B obtained from A by permuting, say y_{ii} and y_{jj}, and by (i), this same permutation is performed on $F(A)$ in order to obtain $F(B)$. Thus by (ii), $g_{ii}(y_{jj})=F(B)_{ii}=F(A)_{jj}=g_{jj}(y_{jj})$. Hence for any j, $g_{jj}(z)=g_{ii}(z)$ for $i=1, \cdots, n$ and $|z-x_{jj}|<\epsilon$ and therefore, since the F_{ii} are analytic, there exists a function $g(z)=g_{11}(z)(=g_{ii}(z),\ i=2, \cdots, n)$, analytic in the open circular domains $|z-x_{jj}|<\epsilon$, $j=1, \cdots, n$, where the x_{jj} are the eigenvalues of Z. Thus, since Z is an arbitrary matrix in D, there exists a function $g(z)$ which is analytic at the eigenvalues of all matrices in D.

In order to show that $g(z)$ satisfies (3.3) we first note, from (3.1), that if F is H-analytic in a neighborhood of a canonical matrix C, then $F(C)$ may be written

$$F(C) = \sum_{i=1}^{k} \sum_{s=0}^{p_i-1} \sum_{t=0}^{p_i-s-1} F(C)_{r_i r_i+t} E_{r_i+s\ r_i+s+t} \tag{3.4}$$

where $r_i = 1 + \sum_{t=1}^{i-1} p_t$ and E_{pq} is the matrix with a 1 in the p, q position and zeros elsewhere.

Now, for each j, $1 \leqq j \leqq k$, let

$$K_{p_j} = \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ & \lambda_j + h_j & 1 & & \cdot \\ & & \lambda_j + 2h_j & \cdot & \cdot \\ & & & \ddots & 1 \\ 0 & & & & \lambda_j + (p_j - 1)h_j \end{pmatrix}$$

then for all $h_j \neq 0$ sufficiently small, F is H-analytic at $K = K_{p_1} \dotplus \cdots \dotplus K_{p_k}$ (since F is H-analytic in a neighborhood of C).

Let $Q_j = (q(j)_{rs})$, $r, s = 1, \cdots, p_j$, where $q(j)_{rs} = 0$ for $r > s$ and $q(j)_{rs} = (-1)^{r+s}/(s-r)!h_j^{s-r}$ for $r \leqq s$, then $Q_j^{-1} = (\tilde{q}(j)_{rs})$ where $\tilde{q}(j)_{rs} = 0$ for $r > s$ and $\tilde{q}(j)_{rs} = 1/(s-r)!h_j^{s-r}$ for $r \leqq s$; also

$$Q_j K_{p_j} Q_j^{-1} = D_{p_j} = \operatorname{diag}(\lambda_j + (i-1)h_j), \qquad i = 1, \cdots, p_j.$$

Now, let $Q = Q_1 \dotplus \cdots \dotplus Q_k$, then $QKQ^{-1} = \Lambda = D_{p_1} \dotplus \cdots \dotplus D_{p_k}$, the canonical form of K. By (i), F is H-analytic at Λ, and as in (3.4),

$$F(\Lambda) = \sum_{i=1}^{n} F(\Lambda)_{ii} E_{ii}.$$

By (i), $F(K) = Q^{-1}F(\Lambda)Q$, therefore, for $0 \leqq i \leqq p_j - 1$, $F(K)_{r_j r_j+i} = \sum_{s=0}^{i} \tilde{q}(j)_{r_j r_j+s} F(\Lambda)_{r_j+s r_j+s} q(j)_{r_j+s r_j+i}$. Thus, by the first part of this proof and the definitions of $q(j)_{rs}$ and $\tilde{q}(j)_{rs}$,

$$\begin{aligned} F(K)_{r_j r_j+i} &= \frac{1}{h_j^i} \sum_{s=0}^{i} \frac{(-1)^{s+i} g(\lambda_j + sh_j)}{s!(i-s)!} \\ &= \frac{1}{i!h_j^i} \sum_{s=0}^{i} (-1)^{s+i} \binom{i}{s} g(\lambda_j + sh_j) = \frac{\Delta^i g(\lambda_j)}{i!h_j^i}. \end{aligned}$$

Since $\lim_{h_j \to 0} \Delta^i g(\lambda_j)/h_j^i = g^{(i)}(\lambda_j)$ [6],

$$\lim_{\Sigma_j |h_j| \to 0} K = C,$$

and the F_{rs} are analytic and therefore continuous in a neighborhood of the components of C, it follows that

$$F(C)_{r_jr_j+i} = \lim_{h_j\to 0} F(K)_{r_jr_j+i} = g^{(i)}(\lambda_j)/i!.$$

Thus (3.3) is proven and hence Theorem 3.1.

It might here be noted that (i) alone is not sufficient for $F(Z)$ to be a primary matric function, as is shown by the function $F(Z) = \sum_{i=1}^{n} F_{ii}E_{ii}$, where $F_{ii} = \sum_{k=1}^{n} z_{kk} = \mathrm{tr}(Z)$. The component functions F_{ij} are analytic functions of the z_{rs} of Z and therefore F is H-analytic; also, for $Y = P^{-1}ZP$, $F(Y) = P^{-1}F(Z)P$. However F_{ii} is not a function of only z_{ii} when Z is a diagonal (or upper triangular) matrix which is necessary for a primary matric function.

It might be further noted, since $F(X)$ is diagonal when X is diagonal, that if X is restricted to the algebra $\mathfrak{D}$ of $n\times n$ diagonal matrices, then $F(X)$ is also a function on $\mathfrak{D}$. Ringleb [5] gave a necessary and sufficient condition for a function to be H-analytic in an algebra; namely, the (analytic) component functions must satisfy a certain set of linear homogeneous partial differential equations of the first order with constant coefficients which depend only on the structure of the algebra. For the algebra $\mathfrak{D}$, this necessary and sufficient condition for a function $F(T) = \sum_{i=1}^{n} F(T)_{ii}E_{ii}$ to be H-analytic in $\mathfrak{D}$ at a matrix $T = \mathrm{diag}(t_{jj})$ is

$$\frac{\partial F(T)_{ii}}{\partial t_{jj}} = 0 \qquad \text{for } i \neq j.$$

Thus hypothesis (ii) of Theorem 3.1 could be restated as follows: Let F also be such that, when restricted to the algebra $\mathfrak{D}$, F is H-analytic in $\mathfrak{D}$ at any diagonal matrix in D.

References

1. W. O. Portmann, *A derivative for Hausdorff-analytic functions*, Proc. Amer. Math. Soc. vol. 10 (1959) pp. 101–105.

2. ———, *Hausdorff-analytic functions of matrices*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 97–101.

3. H. Richter, *Über Matrixfunktionen*, Math. Ann. vol. 122 (1950) pp. 16–34.

4. R. F. Rinehart, *The equivalence of definitions of a matric function*, Amer. Math. Monthly vol. 62 (1955) pp. 395–414.

5. F. Ringleb, *Beitrage zur Funktionentheorie in hypercomplexen Systemen* I, Rend. Circ. Mat. Palermo vol. 57 (1933) pp. 311–340.

6. D. W. Robinson, *An application of the decomposition of a matrix into principal idempotents*, Amer. Math. Monthly vol. 65 (1958) pp. 694–695.

7. H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices*, Blackie and Son Ltd., 1932.

Case Institute of Technology