# CHARACTERIZATIONS OF CONVEX SETS BY LOCAL SUPPORT PROPERTIES

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It is our purpose to establish some new characterizations of convex sets by means of local properties and to derive as a consequence certain known results. This will be done for sets in a topological linear space $L$, such a space being a real linear space with a Hausdorff topology such that the operations of vector addition $x+y$ and scalar multiplication $\alpha x$ are continuous in both variables jointly [**3**]. The principal results are contained in Theorems 4 and 5. In order to describe matters simply, the following notations are used.

NOTATIONS. The interior, closure, boundary and convex hull of a set $S$ in $L$ are denoted by int $S$, cl $S$, bd $S$ and conv $S$ respectively. The closed line segment joining $x \in S$ and $y \in S$ is indicated by $xy$, whereas $L(x, y)$ stands for the line determined by $x$ and $y$. The interior of a set $S$ relative to the minimal linear variety containing it is denoted by intv $S$. Set union, intersection and difference are denoted by $\cup$, $\cdot$ and $\sim$ respectively. Vector addition and subtraction are denoted by $+$ and $-$ respectively. We let 0 and $\phi$ stand for the empty set and the origin of $L$ respectively.

In the statements of theorems and definitions the names of previous authors are indicated for historical purposes.

DEFINITION 1. *Let $S \subset L$. A point $x \in$ bd $S$ is called a point of mild convexity of $S$ if $x$ is not the midpoint of any segment $uv$ with $0 \neq uv \sim x \subset$ int $S$.*

It is desirable to compare this definition with those given by Tietze [**5**] and by Leja and Wilkosz [**4**]. See also Kaufman [**2**]. For a brief summary of earlier results see Bonnesen and Fenchel [**1**, p. 7].

DEFINITION 2. *Let $x \in$ bd $S$, where $S \subset L$. The point $x$ is a point of weak or strong convexity of $S$, or a point of weak or strong concavity of $S$, if there exists a neighborhood $N(x)$ of $x$ and a linear functional $f$ with $f(x) = c$ such that the following conditions hold*:

(a) (Tietze). *The point $x$ is a point of weak convexity of $S$ if $f(y) > c$ with $y \in N(x) \sim x$ implies $y \notin S$. (For strong convexity replace $f(y) > c$ by $f(y) \geqq c$.)*

(b) (Leja and Wilkosz). *The point $x$ is a point of strong concavity of $S$ if $f(y) \leqq c$ with $y \in N(x) \sim x$ implies $y \in S$. (For weak concavity replace $f(y) \leqq c$ by $f(y) < c$.)*

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As observed by Tietze [5], the following theorem of Leja and Wilkosz holds only in $L_2$, the two-dimensional normed linear space.

THEOREM 1 (LEJA AND WILKOSZ). *Each open connected nonconvex set in* $L_2$ *has at least one point of strong concavity* (*see Definition* 2, (b)).

Hence, to remove the restriction to $L_2$, Tietze [5] proved the following theorem for sets in $L_n$, the finite $n$-dimensional case. It also holds in $L$.

THEOREM 2 (TIETZE). *Let $S$ be an open connected set in a topological linear space $L$. If each point of* bd $S$ *is a point of weak convexity of $S$* (*see Definition* 2, (a)), *then $S$ is convex.*

The concept in Definition 1 is a natural one for obtaining a form of the theorem of Leja and Wilkosz [4] for sets in $L$. The following Theorem 3 implies Theorem 2, whereas Theorem 1 implies Theorem 3 *only* for sets in $L_2$, the two-dimensional case.

THEOREM 3. *Let $S$ be an open connected set in a topological linear space $L$, and suppose each point $x \in$* bd $S$ *is a point of mild convexity of $S$* (*see Definition* 1).

*Then $S$ is convex.*

PROOF. Since $S$ is polygonally connected [3], to prove Theorem 3, it is sufficient to prove that $xz \subset S$, $zy \subset S$ implies $xy \subset S$. Let $H_2$ be a two-dimensional plane containing $x$, $y$ and $z$. Then let $K$ be the component of $S \cdot H_2$ containing $x$, $y$ and $z$. Theorem 1, applied to $K$, implies that $K$ is convex, so that $xy \subset S$.

*Note.* A very short proof of Theorem 1 of Leja and Wilkosz exists, and it is given here for completeness. To do this, start as in the above paragraph, so that we merely need to show that $xy \subset S$. Suppose that $xy \not\subset S$. Let $S^*$ denote the set of those boundary points of $S$ which are in the triangle $\operatorname{conv}(x' \cup y \cup z)$ where $x \in \operatorname{intv} x'z$ and $x'z \subset S$. Then $\operatorname{conv} S^* \subset \operatorname{conv}(x' \cup y \cup z)$. It is a very simple matter to show that there exists a point $p \in S^* \cdot \operatorname{conv} S^*$ which is an *exposed* point of conv $S^*$, i.e. there exists a line of support $L_1$ to conv $S^*$ such that $L_1 \cdot \operatorname{conv} S^* = p$. Moreover, one can choose $p$ so that $p \notin x'y$. Since $p \notin x'z \cup zy \subset \operatorname{int} S$, and since $p$ is an exposed point of conv $S^*$, it is trivial to verify that $p$ is a point of strong concavity of $S$ (see Definition 2, (b)).

The following theorem extends Theorem 3 to connected sets without the assumption of openness.

DEFINITION 3. *A hyperplane $H$ strictly separates a set $S$ if each component of the complement of $H$ intersects $S$. The line $L_1$ pierces a set $S$ if* *each hyperplane containing $L_1$ strictly separates $S$.*

THEOREM 4. *Let $S$ be a closed connected set in a topological linear space $L$, with* $\operatorname{int}\operatorname{conv} S \neq 0$. *Assume that each point $x \in \operatorname{bd} S$ is a point of mild convexity of $S$ (see Definition 1). Also assume that each line $L_1$ through $x \in \operatorname{bd} S$ which pierces $S$ contains a segment $xy$ such that $0 \neq \operatorname{intv} xy \subset \operatorname{int} S$.*

*Then $S$ is convex.*

PROOF. Since $\operatorname{int}\operatorname{conv} S \neq 0$, let $u \in \operatorname{int}\operatorname{conv} S$ and $v \in \operatorname{bd} S \sim u$. Clearly each hyperplane containing $uv$ must strictly separate $S$, otherwise $u \in \operatorname{bd}\operatorname{conv} S$. Hence, by hypothesis, $v$ is linearly accessible from $\operatorname{int} S$, so that $\operatorname{int} S \neq 0$. Since $L$ is a topological linear space, each component of $\operatorname{int} S$ is open. Let $K$ be a component of $\operatorname{int} S$. Since each boundary point of $K$ is a boundary point of $S$, the set $K$ satisfies the hypotheses of Theorem 3. Hence, $K$ is convex. Since $L$ is a topological linear space, the $\operatorname{cl} K$ is convex [3]. Suppose that $\operatorname{cl} K \neq S$, and let $y \in S \sim \operatorname{cl} K$. Choose a point $z \in \operatorname{int} K$, and consider a two-dimensional plane $H_2$ containing $zy$. Let $H_2 \cdot K \equiv C$, so that $C$ is a two-dimensional convex body, that is, $\operatorname{intv} C \neq 0$. Since the set of points in $\operatorname{bd} C$ at each of which there exists a unique line of support to $C$ is dense in $\operatorname{bd} C$, there exists a point $x \in \operatorname{bd} C$ (sufficiently close to $yz \cdot \operatorname{bd} C$) through which a unique line of support $L_1$ to $C$ passes which strictly separates $y$ and $z$. Each hyperplane $H$ containing $L_1$ strictly separates $S$. This follows from the fact that $z \notin H$ if and only if $y \notin H$; also if $z \in H$, then $z \in \operatorname{int} S$ implies that $H$ separates $S$. Since $\operatorname{bd} K \subset \operatorname{bd} S$, the hypothesis implies there exists a segment $xp \subset L_1$ such that $0 \neq \operatorname{intv} xp \subset \operatorname{int} S$. Since $(x+p)/2 \in \operatorname{int} S$, relative to $H_2$, there exists a two-dimensional convex set $C_1 \subset H_2$ such that $(x+p)/2 \subset \operatorname{intv} C_1$, and such that $C_1 \subset \operatorname{int} S$. Let $K_1$ be the component of $\operatorname{int} S$ which contains $C_1 \cup \operatorname{intv} xp$. Since, by Theorem 3, each component of $\operatorname{int} S$ is convex, the set $K_1$ is convex, and hence $\operatorname{conv}(C_1 \cup \operatorname{intv} xp) \subset K_1$. However, since $L_1$ is the *unique* line of support to $C$ at $x$, and since $\operatorname{intv} xp \subset K_1 \cdot L_1$, it follows that $K \cdot K_1 \neq 0$. This contradicts the fact that $K$ is a component of $\operatorname{int} S$. Hence, $S = \operatorname{cl} K$, and the theorem is proved.

DEFINITION 4. *Let $S \subset L$ with $p \in S$. The set $S$ is said to have a radius of support relative to $p$ at each of its boundary points uniformly locally if the following holds: For each point $x \in \operatorname{bd} S$ there exists a neighborhood $N(x)$ such that for each point $y \in N(x) \cdot \operatorname{bd} S$ we have $S \cdot R(y, p) \cdot [N(x)+y-x] = 0$, where $R(y, p)$ is the relatively open half-line of the line $L(y, p)$ having endpoint $y$, and not containing $p$, and where $N(x)+y-x$ is the translate of $N(x)$ to the point $y$.*

THEOREM 5. *Let $S$ be a closed connected set in a topological linear space $L$, with* int $S \neq 0$. *Suppose that each point $x \in$* bd *$S$ is a point of mild convexity of $S$ (see Definition* 1). *Also suppose there exists a point $p \in$* int *$S$ such that $S$ has a radius of support relative to $p$ at each of its boundary points uniformly locally.*

*Then $S$ is convex.*

PROOF. In a topological linear space $L$, as a basis of fundamental neighborhoods of the origin $\phi$, it is always possible to restrict oneself to neighborhoods which are *starshaped* and *centrally symmetric* with respect to $\phi$. Since each translate and each nonzero scalar multiple of each neighborhood of $\phi$ is a neighborhood, we may restrict ourselves entirely to such neighborhoods [**3**]. We will do so throughout the following proof.

Let $K$ be that component of int $S$ which contains the point $p$. Since $K$ satisfies the hypotheses of Theorem 3, the set $K$ is convex. Since $S$ is closed, the cl $K$ is a convex subset of $S$, and bd $K \subset$ bd $S$. Suppose that cl $K \neq S$. Then since $S$ is connected, there exists a point $x \in$ bd $K$ which is a limit point of $S \sim$ cl $K$. Without loss of generality, assume that $x$ is the origin $\phi$. This may be accomplished without changing hypotheses by translating $S$ so that $x$ goes to the origin $\phi$. Let $V_1$ and $V_2$ be neighborhoods of $x = \phi$ (centrally symmetric and starshaped relative to $\phi$) such that $V_2 + V_2 \subset V_1$, $V_1 + V_1 \subset N(\phi)$. Since $p \in$ int $K$, we have intv $p\phi \subset$ int $K$. Choose a point $q \in (\text{intv } p\phi) \cdot V_2$. Let $U$ be a neighborhood of $q$ contained in $V_2 \cdot K$ (centrally symmetric and starshaped relative to $q$). Let $V_3 \equiv (U - q) \cdot V_2$, so that $V_3$ is a neighborhood of $\phi$. We have $V_3 \subset V_2$, $V_3 + q \subset V_2$. Since $\phi$ is a limit point of $S \sim$ cl $K$, there exists a point $y \in V_3 \cdot (S \sim \text{cl } K)$. Hence, $z \equiv y + q \in V_3 + q$. Since the segment $qz \subset V_3 + q$, we have $r \equiv py \cdot qz \in V_3 + q$. Since $V_3 + q \subset V_2 \cdot K$, we have $r \in V_2 \cdot K$. Also $y \in V_2 \sim$ cl $K$. Hence, let $ry \cdot (\text{bd } K) \equiv u$. Since $u = \lambda r + (1 - \lambda) y$, where $0 < \lambda < 1$, and since $V_2 + V_2 \subset V_1$, we have $u \in V_1$. Since $-u \in V_1$, $y \in V_1$, we have $-u + y \in V_1 + V_1 \subset N(\phi)$. This implies that $y \in N(\phi) + u$. However, this contradicts the hypothesis that $uy \cdot (N(\phi) + u) \cdot (S \sim u) = 0$, since $y \in S$. Hence, $S = $ cl $K$, and the theorem is proved.

The uniformity of the local radial support property in Theorem 5 cannot be omitted, for consider the following set $S \subset E_2$, where $E_2$ is the Euclidean plane with coordinates $(x_1, x_2)$. Let $S_1 = [(x_1, x_2) : x_1^2 + x_2^2 \leqq 1]$ and $S_2 = [(x_1, x_2) : x_1^2 + (x_2 - 2)^2 = 1]$. Now let $S = S_1 \cup S_2$, $p = (0, 0)$. The set $S$ satisfies all the hypotheses of Theorem 5, except the radial support property is not uniform on account of the point $(0, 1)$.

DEFINITION 5 (*Tietze, see Definition* 2, (a)). *If $x$ is a point of weak convexity of a set $S\subset L$, all those points $y\in L$ for which $f(y)>C$, $y\in N(x)$ form a half-cell of support to $S$ at $x$.*

COROLLARY TO THEOREM 5 (*A generalization of a theorem of Tietze* [5]). I. *Suppose that $S$ is a closed connected set in a topological linear space $L$, and suppose* int $S\neq 0$.

II. *Suppose for each point $x\in$* bd *$S$ there exists a neighborhood $N(x)$ of $x$ such that for each point $y\in N(x)\cdot$* bd *$S$, there exists a half-cell of support of $N(x)+y-x$ to $S$ at $y$.*

*Then $S$ is convex.*

PROOF. Hypothesis II of this corollary implies the last hypotheses of Theorem 5.

REMARK. If in hypothesis II of the corollary we assume that $N(x)=N(\phi)+x$, where $N(\phi)$ is a neighborhood of the origin $\phi$, then we obtain for $L$ the simplest generalization of the theorem of Tietze [5]. If $S$ is locally compact, we may weaken hypothesis II in the the corollary so that it holds on a dense subset of bd $S$, yielding as a result a theorem for sets in $L_n$ of the type studied by Kaufman [2].

## REFERENCES

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