ON THE PRIMITIVITY OF HOPF ALGEBRAS OVER A FIELD WITH PRIME CHARACTERISTIC

EDWARD HALPERN

We recall that an *H-space* consists of a topological space T with a base point $e \in T$ and a (continuous) map $\nabla: T \times T \to T$ such that $\nabla i \simeq I$ and $\nabla j \simeq I$, where i and j are defined by $i(t) = (t, e)$ and $j(t) = (e, t)$, I is the identity map of T, and "$\simeq$" means "homotopic relative to e." The multiplication ∇ is *homotopy-associative* if

$$(1.1) \qquad \nabla(\nabla \times I) \simeq \nabla(I \times \nabla);$$

it is *homotopy-commutative* if

$$(1.2) \qquad \rho\nabla \simeq \nabla$$

where ρ is defined by $\rho(s, t) = (t, s)$, $(s, t \in T)$. We shall assume throughout that T is arcwise connected.

Let H be an associative and anticommutative graded K-algebra with unit 1, where K is a field. We assume throughout that $H^i = 0$ if $i < 0$, and $H^0 = K \cdot 1$. Let H^+ denote the submodule spanned by the elements of positive degree. H is a *Hopf algebra* over K if there is an algebra homomorphism $\Delta: H \to H \otimes H$ (regarding $H \otimes H$ as a graded K-algebra in the usual way) such that

$$\Delta''(x) = \Delta(x) - \Delta'(x) \in H^+ \otimes H^+, \qquad x \in H,$$

where $\Delta': H \to H \otimes H$ is defined by

$$\Delta'(1) = 1 \otimes 1, \qquad \Delta'(x) = x \otimes 1 + 1 \otimes x, \qquad x \in H^+.$$

We shall refer to Δ as the *coproduct.*

The coproduct is *associative* if

$$(1.3) \qquad (\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$$

where I is the identity map of H; it is *anticommutative* if

$$(1.4) \qquad \theta\Delta = \Delta,$$

where θ is defined by

$$\theta(x \otimes y) = (-1)^{ij} y \otimes x, \qquad x \in H^i, y \in H^j.$$

By a *Hopf subalgebra* we mean a graded subalgebra G such that $\Delta(G) \subset G \otimes G$.

Presented to the Society, January 22, 1959; received by the editors April 16, 1959.

It is well-known that the cohomology algebra $H^*(T, K)$, where T is an H-space and K is a field, is a Hopf algebra with coproduct $\Delta = \nabla^*$ (assuming the usual identification given by the Künneth formula). Comparing (1.1) and (1.3), evidently homotopy-associativity of ∇ implies associativity of Δ. It is known that $\theta = \rho^*$; hence, comparing (1.2) and (1.4), we see that homotopy-commutativity of ∇ implies anticommutativity of Δ.

Let (H, Δ) be a Hopf algebra over K. An element $y \in H$ is *primitive* if $\Delta''(y) = 0$. Let $\pi \subset H$ be the subalgebra generated by the primitive elements. It is easy to see that π is a Hopf subalgebra. If $\pi = H$ then we call H a *primitive Hopf algebra*. The following theorem was proved by the author [**3**, Theorem 2.10], and independently by J. C. Moore [**5**].

(1.5) *If H is a Hopf algebra over a field of characteristic zero and the coproduct is associative and anticommutative then H is primitive.*

A simple algebraic example shows that (1.5) is not true in general if the field has prime characteristic. The following theorem is due to H. Samelson [**6**] and J. Leray [**4**]:

(1.6) *Let H be a Hopf algebra over a field and let the coproduct be associative. If H is an exterior algebra generated by odd degree elements then it is primitive.*

The purpose of this paper is to establish primitivity of $H^*(T, Z_p)$ for some H-spaces T, where Z_p is the ring integers modulo a prime p. We shall make use of properties of the Steenrod cohomology operations [**7**] which we denote by

$$St_p^i = \begin{cases} Sq^i & \text{(squares),} \qquad \text{if } p = 2, \\ P_p^i & \text{(reduced powers),} \qquad \text{if } p > 2. \end{cases}$$

Let T be an H-space and suppose $H^*(T, Z_p)$ is a polynomial ring $Z_p[X]$ where $X \subset H^*(T, Z_p)$ and consists of even degree elements if $p \neq 2$. The operations St_p^i are said to *split* on X if for all $i \geqq 0$ and $x \in X$, $St_p^i(x)$ is in the subalgebra generated by x.

THEOREM 1. *Let T be an arcwise connected H-space with homotopy-associative and homotopy-commutative multiplication. If $H^*(T, Z_p) = Z_p[X]$ and the Steenrod cohomology operations split on X then $H^*(T, Z_p)$ is primitive.*

We remark that it then follows on using a Künneth formula that $H^*(T, K)$ is primitive if K is a field of characteristic p.

As an application of Theorem 1 we shall prove the following theorem. For a fixed prime p, a topological space E is *p-elementary* if

$H^*(E, Z_p) \cong Z_p$ or $H^*(E, Z_p) = Z_p[x]$. As examples we cite: The real projective plane ($p \neq 2$), the loop spaces $\Omega(S^{2n+1})$ and complex projective space of infinite dimensions. On the other hand the loop spaces $\Omega(S^{2n})$ are not p-elementary for any p.

THEOREM 2. *Let p be a fixed prime and $E = E_1 \times \cdots \times E_n$, where the E_i are p-elementary spaces. Let T be an arcwise connected H-space with homotopy-associative and homotopy-commutative multiplication. If there is a map $f: T \to E$ or $f: E \to T$ which induces an isomorphism of the cohomology algebras with coefficients in Z_p, then $H^*(T, Z_p)$ is primitive.*

PROOF. Using the Künneth formula and f^* we may represent

$$H^*(T, Z_p) = Z_p[x_1, x_2, \cdots, x_n]$$

where x_i generates $H^*(E_i, Z_p)$ (we may ignore the trivial factors). In view of the Cartan tensor product formula (see [2, Exposé 16, bis 1]) the St_p^i split on $H^*(E_1 \times \cdots \times E_n, Z_p)$ and hence on $\{x_1, x_2, \cdots, x_n\}$ since they commute with f^*. Thus the theorem follows from Theorem 1.

COROLLARY. *A necessary condition that an arcwise connected H-space T with homotopy-associative and homotopy-commutative multiplication be homotopically equivalent to a cartesian product of p-elementary spaces $E_1, E_2, \cdots, E_n$ is that $H^*(T, Z_p)$ be primitive.*

REMARK. Even if the E_i are all H-spaces, the map f is not required to commute with the multiplication in T and the induced multiplication in $E_1 \times E_2 \times \cdots \times E_n$.

2. **The main lemma.** Let $H = K[X]$ be a Hopf algebra over a field K of prime characteristic p. We note:

(2.1) *If $p \neq 2$ each $x \in X$ has even degree.*

(2.2) *We may assume that each $x \in X \cap \pi$ is primitive.*

The first is a well-known consequence of the theorem of A. Borel (see [1, Théorème 6.1]). The second follows from Theorem 2.7 in [3].

We shall assume throughout that X is well-ordered in such a way that if x has lower degree than y then $x < y$. By a *normal monomial* we shall mean a product of the form $M = x_1^{m_1} x_2^{m_2} \cdots x_t^{m_t}$, where the $x_i \in X$ and $x_i < x_{i+1}$. We call $m_1 + \cdots + m_t$ the *length* of M and the number of positive exponents its *width*. If R and S are normal monomials their juxtaposition RS (corresponding to their product as elements of H) is equal to a unique normal monomial which we denote by $\nu(RS)$. If

$$R = x_1^{r_1} x_2^{r_2} \cdots x_t^{r_t}, \qquad S = x_1^{s_1} x_2^{s_2} \cdots x_t^{s_t}, \qquad r_i + s_i = m_i,$$

we define

$$[R, S] = \prod_{i=1}^{t} (r_i, s_i), \qquad m_i!/r_i!s_i! = (r_i, s_i).$$

By induction on width one proves readily

(2.3) *If M is a normal monomial whose factors are primitive then*

$$\Delta(M) = \sum_{\nu(RS)=M} [R, S] R \otimes S,$$

where the summation extends over distinct pairs of normal monomials R, S.

Let $z \in X$ be such that $\Delta''(z) \in \pi \otimes \pi$. Then we may write (uniquely)

$$\Delta''(z) = \sum a(M, N) M \otimes N, \qquad a(M, N) \in K, \tag{2.4}$$

where the summation extends over finitely many distinct pairs of normal monomials M, N in primitive elements of X and the degree of MN is equal to the degree of z. The proof of Theorem 1 will depend on the

MAIN LEMMA. *Let H be a Hopf algebra with an associative and anti-commutative coproduct Δ over a field K with prime characteristic p. Let $H = K[X]$, where $X \subset H$, and let $z \in X$ be such that $\Delta''(z) \in \pi \otimes \pi$. Then there is an element $v \in H$ with the same degree as z such that $z - v \in \pi$ and*

$$\Delta''(v) = \sum \sum a(x^m, x^n) x^m \otimes x^n, \qquad a(x^m, x^n) \in K \tag{2.5}$$

where the outer summation is over (primitive) $x \in X$ and the inner summation is over (positive) m and n with $m+n$ a power of p.

We shall first prove some subsidiary lemmas. It will be convenient to extend the definition of $a(M, N)$ in (2.4) as follows: If Q, R, S, T. are normal monomials in primitive elements of X then

$$a(QR, ST) = a(\nu(QR), \nu(ST)).$$

LEMMA 2.1. *If R, S, T are normal monomials with $R \neq 1$ and $T \neq 1$ then*

$$a(RS, T)[R, S] = a(R, ST)[S, T].$$

PROOF. Since the coproduct is associative we may equate the coefficients of $R \otimes S \otimes T$ in $(\Delta \otimes I)\Delta(z)$ and $(I \otimes \Delta)\Delta(z)$. Since $R \neq 1$ and $T \neq 1$, it is readily seen $(\Delta \otimes I)\Delta'(z)$ and $(I \otimes \Delta)\Delta'(z)$ contribute nothing to these coefficients. We have

$$(\Delta \otimes I)\Delta''(z) = \sum a(M, N)\Delta(M) \otimes N, \tag{2.6}$$

$$(I \otimes \Delta)\Delta''(z) = \sum a(M, N)M \otimes \Delta(N). \tag{2.6$'$}$$

Now using (2.3) it follows that the coefficients of $R\otimes S\otimes T$ in (2.6) and (2.6)$'$, respectively, are

$$a(\nu(RS), T)[R, S] = a(RS, T)[R, S],$$
$$a(R, \nu(ST))[S, T] = a(R, ST)[S, T],$$

and the lemma is proved.

LEMMA 2.2. *If $\nu(MN)$ is of the form w^kQ, where $k \geqq 1$ is the multiplicity of w and $Q \neq 1$ is normal then*

$$a(M, N) = [M, N]a(w^k, Q).$$

PROOF. It suffices to consider M and N as normal monomials of the form x^mR and y^nS respectively, where m and n are the corresponding (positive) multiplicities of x and y and R and S are normal. Note that if $R=1$ and $S=1$ the lemma follows at once from anticommutativity of Δ. Assume that not both $R=1$ and $S=1$; we consider 3 cases:

(i) $x<y$. If $R=1$ the lemma is trivial. If $R\neq 1$ then

$$a(x^mR, N) = [R, N]a(x^m, RN) = [R, N]a(x^m, Q) = [M, N]a(x^m, Q).$$

(ii) $x=y$. If $R=1$ then

$$a(x^m, x^nS) = [x^m, x^n]a(x^{m+n}, Q) = [x^m, N]a(x^{m+n}, Q).$$

If $R\neq 1$ then

$$\begin{aligned} a(x^mR, x^nS) &= [R, N]a(x^m, Rx^nS) \\ &= [R, N]a(x^m, x^nRS) \\ &= [R, N][x^m, x^n]a(x^{m+n}, RS) \\ &= [R, N][x^m, x^n]a(x^{m+n}, Q) \\ &= [x^mR, N]a(x^{m+n}, Q). \end{aligned}$$

(iii) $x>y$. Using case (i) we may write

$$a(N, M) = [N, M]a(y^k, Q).$$

Note that $a(M, N)=a(N, M)$ by anticommutativity of Δ.

LEMMA 2.3. *If $x\in X$, $m+n=r+s$, and*

$$(s-n, n) \not\equiv 0 \pmod{p} \qquad s \geqq n, \tag{2.7}$$

then

$$(r, s)a(x^m, x^n) = (m, n)a(x^r, x^s).$$

PROOF. By Lemma 2.1,

$$(r, m - r)a(x^m, x^n) = (s - n, n)a(x^r, x^s).$$

If we multiply by (m, n) and use the identity

$$(m, n)(r, m - r) = (r, s)(s - n, n),$$

we get

$$(r, s)(s - n, n)a(x^m, y^n) = (m, n)(s - n, n)a(x^r, x^s).$$

In view of (2.7) we may divide out $(s-n, n)$.

LEMMA 2.4. *If $x \in X$ and $m+n=qp^i$, where $q>1$ and $q \not\equiv 0$* (mod p) *then*

(2.8) $a(x^m, x^n) = 0$ *if p^i does not divide m and n,*

(2.9) $a(x^{rp^i}, x^{sp^i}) = (r, s)a(x^{(q-1)p^i}, x^{p^i})/q \qquad$ *if $s \not\equiv 0$* (mod p).

PROOF OF (2.8). Suppose $n<(q-1)p^i$. Since n is not divisible by p^i,

$$((q - 1)p^i - n, n) \not\equiv 0 \qquad \pmod{p};$$

hence by Lemma 2.3,

$$qa(x^m, x^n) = (m, n)a(x^{p^i}, x^{(q-1)p^i}).$$

Since m and n are not divisible by p^i, $(m, n) \equiv 0$ (mod p) and (2.8) follows. If $n>(q-1)p^i$ then $m<(q-1)p^i$ and hence $a(x^n, x^m)=0$. By anticommutativity of Δ, (2.8) follows.

PROOF OF (2.9). Note that

$$(sp^i - p^i, p^i) = ((s - 1)p^i, p^i) \equiv (s - 1, 1) = s \not\equiv 0, \qquad \pmod{p}.$$

Therefore (2.9) is obtained on applying Lemma 2.3.

PROOF OF THE MAIN LEMMA. Let V be a normal monomial composed of primitive factors and of the same degree as z. We consider two types of V:

(i) V has width greater than 1. Then we may write $V=x^rS$, where x is the first factor of V and its multiplicity is $r \geqq 1$, and $S \neq 1$. Put $a(V)=a(x^r, S)$. Then using (2.3) and Lemma 2.2, we may write

$$\begin{aligned}\Delta''(a(V)V) &= \sum_{V=\nu(MN);M\neq 1,N\neq 1} a(V)[M, N]M \otimes N \\ &= \sum_{V=\nu(MN);M\neq 1,N\neq 1} a(M, N)M \otimes N.\end{aligned}$$

It follows that if $M\neq 1$, $N\neq 1$, and $\nu(MN)=V$ then $M\otimes N$ has zero coefficient in $\Delta''(z-a(V)V)$.

(ii) $V=x^{qp^i}$, where $q>1$ and $q\not\equiv 0 \pmod p$. Put

$$a(V) = a(x^{(q-1)p^i}, x^{p^i})/q.$$

Using (2.3) we may write

$$\Delta''(a(V)V) = \sum_{r,s>0;r+s=q} a(V)(r, s)x^{rp^i} \otimes x^{sp^i}.$$

Applying (2.9) to the terms for which $s\not\equiv 0$ we may write

$$\Delta''(a(V)V) = \sum_{s\not\equiv 0} a(x^{rp^i}, x^{sp^i})x^{rp^i} \otimes x^{sp^i} + \sum_{s\equiv 0} a(V)(r, s)x^{rp^i} \otimes x^{sp^i}. \tag{2.10}$$

We assert that if $m+n=qp^i$ then $x^m\otimes x^n$ has zero coefficient in $\Delta''(z-a(V)V)$. In view of (2.8) only terms with m and n both divisible by p^i can occur. In view of (2.10) only terms with

$$m = rp^i, \quad n = sp^i, \quad r + s = q, \quad r > 0, \quad s > 0, \quad s \equiv 0 \pmod p$$

can occur. But $s\equiv 0$ and $q\not\equiv 0$ imply $r\not\equiv 0$. Thus, since Δ is anticommutative,

$$a(x^{rp^i}, x^{sp^i}) = a(x^{sp^i}, x^{rp^i}) = 0,$$

and the assertion is proved.

Now define

$$v = z - \sum a(V)V$$

where the summation extends over all V of types (i) and (ii). Then v evidently has the properties asserted in the main lemma.

3. **Proof of Theorem** 1. Let $H^*(T, Z_p)=Z_p[X]$; assume that the elements of $X\cap\pi$ are primitive (see (2.2)). If $H^*(T, Z_p)$ is not primitive then there is an element $z\in X$ which is not in π. Moreover, if we take z of lowest degree then $\Delta''(z)\in\pi\otimes\pi$ and we may write (2.4). Since $\Delta=\nabla^*$ is associative and anticommutative, there is an element $v\in H$ with the properties specified by the main lemma. We shall show that v is primitive; this will produce a contradiction for it implies that $z\in\pi$.

We shall make use of the following properties of St_p^i:

$$St_p^i: H^q(T, Z_p) \to H^{q+r(p-1)}(T, Z_p) \tag{3.1}$$

where $r=i$ if $p=2$ and $r=2i$ if $p\neq 2$.

$$St_p^i\Delta = \Delta St_p^i, \tag{3.2}$$

where

$$St_p^i(u \otimes w) = \sum_{i=j+k} St_p^j(u) \otimes St_p^k(w). \tag{3.3}$$

$$St_p^i(u) = \begin{cases} u, & \text{if } i = 0, \\ u^p, & \text{if } r = \text{degree of } u, \\ 0, & \text{if } r > \text{degree of } u, \end{cases} \tag{3.4}$$

where r is as defined above.

From (3.3) and (3.4) it follows that St_p^i commutes with Δ' and hence also with Δ'' in view of (3.2). Thus

$$St_p^i\Delta''(v) = St_p^i\Delta''(v - z) + \Delta''St_p^i(z). \tag{3.5}$$

Now consider the expression (2.5) for $\Delta''(v)$. Let $a(x^m, x^n)x^m \otimes x^n$ be a summand such that md is maximum, where d is the degree of x. Put

$$a = a(x^m, x^n), \qquad m + n = p^k.$$

In (3.5) take $i = mj$, where $j = d$ if $p = 2$, and $2j = d$ if $p \neq 2$. We shall prove:

A. $St_p^{mj}(z) = 0$.

B. The coefficient of $x^{mp} \otimes x^n$ in $St_p^{mj}\Delta''(v)$ is $a(x^m, x^n)$.

C. The coefficient of $x^{mp} \otimes x^n$ in $St_p^{mj}\Delta''(v - z)$ is zero.

In view of (3.5) it follows from A, B, C that $a(x^m, x^n) = 0$, and hence v is primitive.

Proof of A. The degrees of z and $St_p^{mj}(z)$ are dp^k and $d(p^k + m(p-1))$, respectively. The latter is not a multiple of the former since $p^k > m$, $p - 1$. Thus A follows from the fact that $St_p^{mj}(z)$ is in the subalgebra generated by z.

Proof of B. We have

$$St_p^{mj}(ax^m \otimes x^n) = ax^{mp} \otimes x^n + \sum u_i \otimes w_i \tag{3.6}$$

where the degrees of the u_i are less than mdp. It remains to show that no other summand $by^r \otimes y^s$ in $\Delta''(v)$ can contribute to the coefficient of $x^{mp} \otimes x^n$. If the degree of y^r is less than md this is clear; if the degree of y^r is md then, writing a similar expression to (3.6) for $St_p^{mj}(by^r \otimes y^s)$, we see that only $by^{rp} \otimes y^s$ has the same bidegree as $x^{mp} \otimes x^n$. But if $y^r \otimes y^s \neq x^m \otimes x^n$ then $y \neq x$ or $r \neq m$, and hence $St_p^{mj}(y^r \otimes y^s)$ contributes nothing to the coefficient of $x^{mp} \otimes x^n$.

Proof of C. Combining (2.4) and (2.5) we may write

$$\Delta''(v - z) = -\sum a(M, N)M \otimes N; \tag{3.7}$$

note that $M\otimes N$ has the property that $MN\neq y^{p^i}$ for $y\in X$. Consider such a term $M\otimes N$. Let (d_1, d_2) be its bidegree, and $c_{M,N}$ the coefficient of $x^{mp}\otimes x^n$ in $St_p^{mj}(M\otimes N)$. If $d_1<md$ then it is clear that $c_{M,N}=0$. If $d_1=md$ then the only term in $St_p^{mj}(M\otimes N)$ with the same bidegree as $x^{mp}\otimes x^n$ is $M^p\otimes N$. In view of the restriction on MN, $M^p\otimes N\neq x^{mp}\otimes x^n$, and hence $c_{M,N}=0$. Finally, we complete the proof of C and hence of Theorem 1 by showing that if $d_1>md$ then $a(M, N)=0$.

Let $M\otimes N$ be such that d_1 is maximum. In (3.5) take $i=d_1$ if $p=2$, and $i=d_1/2$ if $p\neq 2$ (the latter is possible since if $p\neq 2$, M has even degree by (2.1)). We assert:

A′. $St_p^i(z)=0$.

B′. The coefficient of $M^p\otimes N$ in $St_p^i\Delta''(v-z)$ is $-a(M, N)$.

C′. The coefficient of $M^p\otimes N$ in $St_p^i\Delta''(v)$ is zero.

In view of (3.5), A′, B′, C′ imply $a(M, N)=0$. The proof of C′ follows immediately from $md<d_1$. For if (e_1, e_2) is the bidegree of a term in $St_p^i\Delta''(v)$ then e_1 is at most $md+d_1(p-1)<pd_1$. The proof of B′ is very similar to the proof of B and we omit the details. To prove A′ it suffices to show that the degree of $St_p^i(z)$ which is $dp^k+d_1(p-1)$ is not a multiple of dp^k (the degree of z) or, equivalently, that $d_1(p-1)$ is not a multiple of dp^k.

Consider $a(x^m, x^n)$ again and put $m=qp^i$, where $q\not\equiv 0 \pmod p$. By Lemma 2.1, we have

$$qa(x^m, x^n) = (p^k - p^i - n, n)a(x^{p^i}, x^{p^k-p^i}).$$

Thus if $a(x^m, x^n)\neq 0$ then $a(x^{p^i}, x^{p^k-p^i})\neq 0$. By anticommutativity of Δ, then $a(x^{p^k-p^i}, x^{p^i})\neq 0$. Since the term $x^m\otimes x^n$ was chosen so that md was maximum, it follows that

$$md \geqq (p^k - p^i)d \geqq (p^k - p^{k-1})d.$$

Combining this with the inequalities

$$dp^k > d_1 > md$$

and multiplying through by $(p-1)/dp^k$ gives

$$(p - 1) > \frac{d_1(p - 1)}{dp^k} > \left(1 - \frac{1}{p}\right)(p - 1).$$

Thus $d_1(p-1)/dp^k$ is not an integer.

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UNIVERSITY OF MICHIGAN

SOME GLOBAL PROPERTIES OF HYPERSURFACES[1]

ROBERT E. STONG

1. **Introduction.** The translation theorem of Hopf [1] has been extended by Hsiung [2] and Voss [4] independently to hypersurfaces and by Hsü [3] to other elementary transformations. The purpose of this paper is to extend to hypersurfaces in $(n+1)$-dimensional Euclidean space some results obtained by Hsü [3] for the case $n=2$.

All hypersurfaces mentioned will be assumed to be twice differentiably imbedded in an $(n+1)$-dimensional Euclidean space $E^{n+1}(n+1\geqq 3)$. The notation used will be that of Hsiung [2]. In particular, X, N, M_1, A denote the position vector, unit inner normal, first mean curvature, and area for the hypersurface V^n. Corresponding quantities for other hypersurfaces will be denoted by *, or by primes.

Considerable use will be made of the vector product defined by Hsiung [2]. Namely, if $i_1, \cdots, i_{n+1}$ denotes a fixed frame of mutually orthogonal unit vectors and $A_1, \cdots, A_n$ are n vectors whose components in this frame are A_i^α $(i=1, \cdots, n;\ \alpha=1, \cdots, n+1)$, the vector product is defined by

$$A_1 \times \cdots \times A_n = (-1)^n \begin{vmatrix} i_1 & i_2 & \cdots & i_{n+1} \\ \overset{1}{A}_1 & \overset{2}{A}_1 & \cdots & \overset{n+1}{A}_1 \\ \cdot & \cdot & \cdot & \cdot \\ \overset{1}{A}_n & \overset{2}{A}_n & \cdots & \overset{n+1}{A}_n \end{vmatrix}.$$

Presented to the Society, June 20, 1959; received by the editors April 20, 1959.

[1] This is a portion of a master's thesis at the University of Oklahoma directed by Professor T. K. Pan.