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A COUNTABLE INTERPOLATION PROBLEM

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1. Let $\mathcal{H}$ be the set of all order-preserving homeomorphisms of $I=[0, 1]$ onto itself. $\mathcal{H}$ is a metric space in the uniform metric ρ:

$$\rho(f_1, f_2) = \max_I |f_1(x) - f_2(x)|, \qquad f_1, f_2 \in \mathcal{H}. \tag{1}$$

Franklin [1] has proved the following theorem: (A) Let A and B be two countable sets, each dense on I. Then the set of analytic $f\in\mathcal{H}$, such that $f(A)=B$, is dense in $\mathcal{H}$.

It follows from (A) and from its extension in [2] that there exist nontrivial analytic functions $f\in\mathcal{H}$, such that $f(x)$ is transcendental for each transcendental $x\in I$, and for each algebraic $x\in I$, x and $f(x)$ are algebraic and of the same degree.

In this note, without using either of these results, we prove a similar but complementary statement by means of Baire's Category Theorem.

THEOREM 1. *Let $\mathcal{H}_\alpha$, $\alpha>2$, be the subset of $\mathcal{H}$ consisting of all functions $f\in\mathcal{H}$, whose values are either rational or transcendental and approximable to degree $>\alpha$, for each algebraic $x\in I$. Then $\mathcal{H}_\alpha$ is a dense G_δ-set of second category in $\mathcal{H}$.*

2. Since $\mathcal{H}$ is not complete in ρ, we first remetrize it. Let

$$\sigma(f_1, f_2) = \rho(f_1, f_2) + \rho(f_1^{-1}, f_2^{-1}), \qquad f_1, f_2 \in \mathcal{H}. \tag{2}$$

LEMMA 1. *$\mathcal{H}$ is complete in the σ-metric.*

Let $\mathfrak{F}=I^I$ be the set of all continuous maps from I into I, then $\mathfrak{F}$ is complete in ρ. Let $\{f_n\}$, $n=1, 2, \cdots$, be a σ-Cauchy sequence in $\mathcal{H}$. Then $\{f_n\}$ is also a ρ-Cauchy sequence in $\mathfrak{F}$, therefore $f_n\rightarrow f$,

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$f\in\mathfrak{F}$. Similarly, $\{f_n^{-1}\}$ is also a ρ-Cauchy sequence in $\mathfrak{F}$, therefore $f_n^{-1}\to h$, $h\in\mathfrak{F}$. Finally, $f[h(x)]=h[f(x)]=x$, therefore $f\in\mathfrak{F}$ and $h=f^{-1}\in\mathfrak{F}$. Hence $f\in\mathfrak{H}$ and $\mathfrak{H}$ is complete.

Henceforth we consider $\mathfrak{H}$ in the σ-metric only.

LEMMA 2. *Let G be an open interval on a line, with rational endpoints. Let $A=\{a_n\}$, $n=1, 2, \cdots$, be the sequence of all rational members of G, and put $a_n=p_n/q_n$ where p_n and q_n are rational integers, $q_n>0$, and g.c.d. $(p_n, q_n)=1$. Let $\alpha>2$ and $\epsilon>0$. Put*

$$(3)\qquad \begin{aligned} A(\epsilon, \alpha) &= \bigcup_{n=1}^{\infty} (p_n/q_n - \epsilon/q_n^{\alpha}, p_n/q_n + \epsilon/q_n^{\alpha}), && 0 \notin G,\\ A(\epsilon, \alpha) &= (-\epsilon, \epsilon) \cup \bigcup_{n=1}^{\infty} (p_n/q_n - \epsilon/q_n^{\alpha}, p_n/q_n + \epsilon/q_n^{\alpha}), && 0 \in G.\end{aligned}$$

Let $\{\epsilon_m\}$, $m=1, 2, \cdots$, be a sequence of positive numbers decreasing steadily to 0. Let

$$(4)\qquad A_\alpha = \bigcap_{m=1}^{\infty} A(\epsilon_m, \alpha).$$

Then $x\in A_\alpha$ if and only if either (1) $x\in A$ or (2) $x\in G$, x is a transcendental number approximable to degree $>\alpha$.

It is clear that $A\subset A_\alpha$. Let $y\in G$ be a real number approximable to degree $d>\alpha$. By the definition of this concept [**3**; **4**] this means that the equation

$$|y - p/q| < 1/q^d$$

has infinitely many solutions in rational integers p, q, g.c.d. $(p, q)=1$. This implies that $y\in A(\epsilon_m, \alpha)$ for every m, and so $y\in A_\alpha$. Since $d>2$, y is transcendental by Roth's theorem [**5**].

Suppose now that $y\in A_\alpha$, y irrational. By the definition of A_α this means that the inequality

$$p/q - \epsilon/q^\alpha < y < p/q + \epsilon/q^\alpha$$

is satisfied in rational integers p, q, g.c.d. $(p, q)=1$, for arbitrarily small ϵ. Since y is irrational there are infinitely many distinct such solutions. This shows that y is approximable to degree $>\alpha$ and therefore transcendental.

3. We now prove Theorem 1.

Let $R=\{r_n\}$, $B=\{b_n\}$, $n=1, 2, \cdots$, be the sets of all rational and all algebraic numbers in (0, 1) respectively, taken in the above enumerations. Sets $R(\epsilon_m, \alpha)$ are defined as in (3). Put

$$\mathfrak{U}_{n,m} = \{f \mid f \in \mathcal{H}, f(b_n) \in R(\epsilon_m, \alpha)\}. \tag{5}$$

Since $R(\epsilon_m, \alpha)$ is an open set dense on I, it follows that $\mathfrak{U}_{n,m}$ is, for each n, m, an open set dense in $\mathcal{H}$. By Baire's Theorem [6] the set

$$\mathfrak{U} = \bigcap_{n,m=1}^{\infty} \mathfrak{U}_{n,m}$$

is therefore a dense G_δ-set of second category in $\mathcal{H}$.

Since $f \in \mathfrak{U}$ if and only if

$$f(b_n) \in \bigcap_{m=1}^{\infty} R(\epsilon_m, \alpha), \qquad n = 1, 2, \cdots$$

it follows from Lemma 2 that $f \in \mathfrak{U}$ if and only if for each n $f(b_n)$ is either rational or transcendental and approximable to degree $>\alpha$. Therefore $\mathfrak{U} = \mathcal{K}_\alpha$ and the theorem is proved.

COROLLARY. *Theorem* 1 *remains true if "approximable to degree* $>\alpha$*" is replaced by "a Liouville number."*

A number x is a Liouville number [3; 4] if it is approximable to any degree. To prove the corollary it suffices to take a sequence $\{\alpha_m\}$, $m = 1, 2, \cdots$, of real numbers increasing steadily to infinity and with $\alpha_1 > 2$. We then consider the set

$$\mathcal{K}_\infty = \bigcap_{m=1}^{\infty} \mathcal{K}_{\alpha_m},$$

which is a dense G_δ-set of second category in $\mathcal{H}$ since each $\mathcal{K}_{\alpha_m}$ is such a set.

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