A SUFFICIENT CONDITION THAT A MAPPING OF RIEMANNIAN MANIFOLDS BE A FIBRE BUNDLE

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1. Introduction. All manifolds, maps, tensor-fields, curves, etc. will be of differentiability class C^∞ unless mentioned otherwise. If x is a point of a manifold X , X_x denotes the tangent space at x . If $\phi: X \rightarrow B$ is a C^∞ map of manifolds, $\phi_*: X_x \rightarrow B_{\phi(x)}$ is the linear map on tangent vectors induced by ϕ . We suppose

- (a) ϕ is of maximal rank on X , i.e. $\phi_*(X_x) = B_{\phi(x)}$ for all $x \in X$.
 (1.1) (b) X and B are Riemannian manifolds, and the isomorphism $\phi_*: X_x / \phi_*^{-1}(0) \approx B_{\phi(x)}$

preserves the inner-products defined by the metrics on these spaces, for all x in X .

The main theorem is:

THEOREM 1. *If X is complete as a Riemannian space, so is B . ϕ is then a locally trivial fibre space. If in addition the fibres of ϕ are totally geodesic submanifolds of X , ϕ is a fibre bundle with structure group the Lie group of isometries of the fibre.*

2. Generalities [2; 1]. If $x \in X$, $v_1, v_2 \in X_x$, let (v_1, v_2) denote the inner product that defines the Riemannian metric. If $\sigma: [0, 1] \rightarrow X$ is a curve, let $\sigma': t \rightarrow \sigma'(t)$ denote the tangent vector field to σ . If $v: t \rightarrow v(t) \in X_{\sigma(t)}$ is a vector-field along σ , let Δv denote the covariant derivative of v along σ [1].

Let us recall how it is defined by Elie Cartan's method of orthonormal moving frames: Suppose U is an open set of X and w_i ($1 \leq i, j, k \dots \leq n = \dim X$, summation convention) 1-differential forms in U with

$$ds^2 = w_i \cdot w_i \text{ (Symmetric Product).}$$

There are unique 1-forms w_{ij} in U , called the *connection forms*, such that

$$dw_i = w_{ij} \wedge w_j \text{ (Exterior Product),}$$

$$w_{ij} = -w_{ji}.$$

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Then, for the portion of σ in U , $\Delta v(t) \in X_{\sigma(t)}$ is defined by the condition:

$$(2.1) \quad w_i(\Delta v(t)) = \frac{d}{dt} w_i(v(t)) - w_{ij}(\sigma'(t)) w_j(v(t)).$$

Suppose now that σ_s is a one-parameter family of curves, a deformation of σ , i.e. $\sigma_0 = \sigma$, and v is the corresponding infinitesimal deformation field, i.e. $v(t)$ is the tangent vector to the curve $s \rightarrow \sigma_s(t)$ at $s=0$. Put $f(s) = \text{length of the curve } \sigma_s$, and suppose each curve σ_s is parametrized proportionally to arc-length. The formula for the "first variation" of arc length is then:

$$(2.2) \quad \begin{aligned} f'(0) &= \int_0^1 (\sigma'(t), \Delta v(t)) / f(0) dt \\ &= (\sigma'(t), v(t)) / f(0) \Big|_{t=0}^{t=1} - \int_0^1 (\Delta \sigma'(t), v(t)) / f(0) dt. \end{aligned}$$

The condition that σ be a geodesic is then

$$(2.3) \quad \Delta \sigma'(t) = 0.$$

3. Proof of Theorem 1. Let us return now to the map $\phi: X \rightarrow B$ of §1. For $x \in X$, a $v \in X_x$ is said to be *vertical* if $\phi_*(v) = 0$. The vectors in the orthogonal complement to the vertical vectors are called *horizontal*. P_V and P_H denote the projections of X_x on these subspaces. The field $x \rightarrow P_V(X_x)$ of tangent subspaces defines, with the metric, a "bundle-like" foliation on X [7]. A curve in X is said to be horizontal if its tangent vector field is horizontal. If $\sigma: [0, 1] \rightarrow B$ is a curve, $x_0 \in \phi^{-1}(\sigma(0))$, there is at most one horizontal lifting $\sigma_1: [0, 1] \rightarrow X$ of σ beginning at x_0 , and horizontal liftings always exist *locally* [4]. If they always exist globally as well, the field of horizontal subspaces is a connection for the map ϕ in the sense of Ehresmann [4], and the map ϕ is a locally trivial fibre space.

PROPOSITION 3.1. *If $\sigma: [0, 1] \rightarrow X$ is a curve, v a vector field along σ , $\phi\sigma: t \rightarrow \phi\sigma(t)$ and $\phi_*v: t \rightarrow \phi_*v(t)$ the projections in B , then*

$$(3.1) \quad \Delta \phi_*(v) = \phi_*(\Delta(P_H(v))).$$

Hence in particular the projection of a horizontal geodesic of X is a geodesic of B , and conversely the horizontal lifting of a geodesic of B , if it exists, is a geodesic.

PROOF. Suppose $\dim B = n - m$. Adopt the following ranges of in-

dices and summation conventions: $1 \leq a, b, c, \leq m$; $m+1 \leq u, v, \dots \leq n$. Let θ_u be 1-forms in an open set U of B with

$$ds^2 = \theta_u \cdot \theta_u.$$

Suppose θ_{uv} are the corresponding connection forms. Let w_i be 1-forms in an open set U' of X with $\phi(U') = U$ and $ds^2 = w_i \cdot w_i$, $w_u =$ the one form on U' induced by ϕ from θ_u , to be denoted here by $\phi^*(\theta_u)$.

Suppose $\theta_{uv} = f_{uvw}\theta_w$, $w_{ij} = g_{ijk}w_k$, where w_{ij} are the connection forms corresponding to w_i in U' . One derives by calculation:

$$(3.2) \quad \begin{aligned} (a) \quad g_{uvw} &= \phi^*(f_{uvw}), \\ (b) \quad g_{uav} &= g_{uva}, \\ (c) \quad g_{uab} &= g_{uba}. \end{aligned}$$

(3.2a) now implies 3.1, via 2.1.

PROPOSITION 3.2. *If X is complete as a Riemannian metric space, so is B . In this case the horizontal liftings of paths² of B exist globally, i.e. ϕ is a fibre space.*

PROOF. The horizontal lifting of a small geodesic segment of B is a geodesic segment. If X is complete it can be completely extended to a geodesic of infinite length. The projection in B of the extension is an extension of the given segment to an infinite geodesic, hence every geodesic segment of B can be indefinitely extended.

To prove the second part, note that the horizontal lifting of a path of B has the same length. If $\sigma: [0, 1] \rightarrow B$ is a path, local horizontal liftings always exist; if we try to get a global lifting by the usual process of continuation we run into no obstruction since the partial liftings will always lie in a fixed bounded, hence compact region of X .

We assume from now on that every path in B has horizontal liftings starting at arbitrary points of the initial fibre. If σ is a path: $[0, 1] \rightarrow B$, there is a diffeomorphism $h_\sigma: \phi^{-1}(\sigma(0)) \rightarrow \phi^{-1}(\sigma(1))$ obtained by mapping each $x_0 \in \phi^{-1}(\sigma(0))$ into the end-point of the horizontal lifting of σ starting at x_0 [4].

PROPOSITION 3.3. *If all the fibres of ϕ are totally geodesic submanifolds of X , then for each path σ of B , h_σ is an isometry of $\phi^{-1}(\sigma(0))$ onto $\phi^{-1}(\sigma(1))$.*

PROOF. We use the first variation formula 2.2. Suppose $\alpha: [0, 1] \rightarrow X$

² A path is a continuous image of $[0, 1]$ that is composed of a finite number of unbroken C^∞ curves.

is a curve completely on a fibre of ϕ , i.e. a vertical curve. Suppose we deform α in a horizontal direction i.e. α_s is a one-parameter family of curves, with infinitesimal deformation field $v: t \rightarrow v(t) \in X_{\alpha(t)}$ a *horizontal* curve. If $f(s) = \text{length of } \alpha_s$, 2.2 says that $f'(0) = 0$, since $\Delta\alpha'$ is a *vertical* vector field (because the fibre is totally geodesic).

If $\sigma: [0, 1] \rightarrow B$ is a curve, the image curve of a curve in $\phi^{-1}(\sigma(0))$ by h is obtained by deforming along horizontal curves covering σ , hence has the same length. Q.E.D.

We assume then that the fibres of ϕ are totally geodesic. Fix a $b_0 \in B$, and set $F = \phi^{-1}(b_0)$. Let G be the Lie group of isometries of F . If $b \in B$, let G_b be the set of isometries of F onto $\phi^{-1}(b)$. Let E be the point-set union of all G_b , for $b \in B$. Let $p: E \rightarrow B$ be the map $G_b \rightarrow b$. G acts in an obvious way on E on the left, and E would be a principal fibre bundle with G as structure group, except that so far it has no topology or manifold structure. Suppose though one can find a locally finite open covering of B , $\{U\}$, such that in each open set U of the covering there is a cross-section $f_U: U \rightarrow E$. Then, for each U and U' of the covering that intersect, there is a map $g_{UU'}: U \cap U' \rightarrow G$ such that

$$f_{U'}(b) = g_{U'U}(b)f_U(b) \quad \text{for } b \in U \cap U'.$$

Suppose that the covering and the cross-sections can be chosen so that all the maps $g_{U'U}$ are C^∞ ; clearly then they define a C^∞ coordinate bundle in the sense of Steenrod [8] which is isomorphic to E , and one can use this isomorphism to define E as a C^∞ manifold.

We use Proposition 3.3 to define such cross-sections. Choose the covering so that each U is a geodesically convex open set of B , and pick a point b_U in U and a curve σ_U joining b_0 to each b_U . Any $b \in U$ is on a unique geodesic with b_U , and one obtains a path $\sigma_{b,U}$ depending on b and U joining b_0 with b . Define $f_U(b) = h_{\sigma_{b,U}} \in B_b$. It is easy to see that the transition functions $g_{UU'}: U \cap U' \rightarrow G$ are C^∞ ; they are defined by solutions of C^∞ ordinary differential equations depending in a C^∞ way on initial conditions.

Having defined E as a C^∞ principal fibre bundle with structure group G , it is clear that the given fibre space $\phi: X \rightarrow B$ is isomorphic to the associated bundle with fibre F . The proof of Theorem 1 is then finished.

Let Ω be the space of loops of B starting at b_0 . $h: \sigma \rightarrow h_\sigma$ is then a map $\Omega \rightarrow G$. Let B_G be the base of the universal classifying bundle for G . The bundle E is induced by a map, the classifying map, $B \rightarrow B_G$. The loops of B_G have the same homotopy type as G , and it is clear that, up to homotopy, h is the same as the map $\Omega \rightarrow \Omega(B_G)$ induced by the classifying map [6].

4. Some examples. First, note that the homogeneous spaces of Lie groups with compact isotropy groups provide a class of examples satisfying the hypotheses of Theorem 1. Of course, Theorem 1 is well-known for these examples [8]. This suggests as generalization the mappings $\phi: X \rightarrow B$ which *locally* have the structure of principal fibre bundle with a Cartan connection [5]. Recall that the bundle of orthonormal frames of a Riemannian manifold has this sort of structure.

To describe these examples, one should be given

(1) a map $\phi: X \rightarrow B$ of manifolds which is of maximal rank everywhere, with $\dim X = n$, $\dim B = n - m$,

(2) a real Lie algebra of dimension n , $\mathfrak{G}$, a subalgebra $\mathfrak{K}$ of dimension m , a subspace $\mathfrak{M}$ of $\mathfrak{G}$ with $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{M}$, $[\mathfrak{K}, \mathfrak{M}] \subset \mathfrak{M}$, a positive definite bilinear form on $\mathfrak{M}$ with respect to which the operations of $\text{Ad } \mathfrak{K}$ are skew-symmetric. (We follow Chevalley [3] for the notations of Lie algebra theory.) Choose a basis $\mathfrak{g}_i$ of $\mathfrak{G}$ such that (a) the $\mathfrak{g}_a$ span $\mathfrak{K}$ and the $\mathfrak{g}_u$ span $\mathfrak{M}$ (same range of indices as in §3), (b) the $\mathfrak{g}_u$ are an orthonormal basis with respect to the bilinear form on $\mathfrak{M}$. Let $[\mathfrak{g}_i, \mathfrak{g}_j] = c_{ijk}\mathfrak{g}_k$ be the structural equations of $\mathfrak{G}$ with respect to this basis.

(3) n vector-fields θ_i for X which are linearly independent at every point of X , i.e. form a *parallelism* for X , such that

(a) $[\theta_a, \theta_i] = c_{aij}\theta_j$, i.e. only part of the structural equations are satisfied, and

(b) for $x \in X$, the $\theta_a(x)$ form a basis for the vectors tangent to the fibre of ϕ through x .

One then defines a Riemannian metric for X by the requirement that, for $x \in X$, the $\theta_i(x)$ form an orthonormal basis for X_x , i.e. if w_i are the dual basis of differential forms of X , $w_i(\theta_j) = \delta_{ij}$, then

$$ds^2 = w_i \cdot w_i.$$

Call the vectors spanned by the θ_a the *vertical vectors* and let P_H and P_V be the projections defined on X_x , as before.

PROPOSITION 4.1. *With this metric for X , if the fibres of ϕ are connected B inherits a Riemannian metric such that, for each x , the isomorphism $\phi_*: P_H(X_x) \approx B_{\phi(x)}$ preserves inner products and the fibres of ϕ are totally geodesic.*

PROOF. Suppose $[\theta_i, \theta_j] = f_{ijk}\theta_k$, where the f_{ijk} are real-valued C^∞ functions on X , $f_{ajk} = c_{ajk}$. Then [3, p. 154]

$$(4.1) \quad dw_i = \frac{1}{2} f_{jik} w_j \wedge w_k.$$

Suppose $w_{ij} = h_{ijk}w_k$ are the connection forms for the Riemannian metric, i.e. $dw_i = w_{ij} \wedge w_j$, $w_{ij} = -w_{ji}$.

One calculates:

$$(4.2) \quad h_{ijk} = \frac{1}{2} (f_{kji} - f_{jik} + f_{ikj})$$

Then, if $\sigma: [0, 1] \rightarrow K$ is a curve, and $v: t \rightarrow v(t) \in X_{\sigma(t)}$ is a vector field along σ ,

$$(4.3) \quad w_i(\Delta v(t)) = \frac{d}{dt} w_i(v(t)) - h_{ijk}(\sigma'(t))w_k(\sigma'(t))w_j(v(t)).$$

Now, suppose

$$S: (s, t) \rightarrow S(s, t), \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 1$$

is a surface in X such that: for fixed s , $t \rightarrow S(s, t)$ is vertical, i.e. lies in a fibre, and for fixed t , $s \rightarrow S(s, t)$ is horizontal.

Suppose $S^*(w_i) = A_i(s, t)dt + B_i(s, t)ds$. The geometric interpretation of these functions is that $A_i\theta_i(S(s, t))$ resp. $B_i\theta_i(S(s, t))$ is the tangent vector to the curve $t \rightarrow S(s, t)$ resp. $s \rightarrow S(s, t)$. Then

$$(4.4) \quad A_u = 0, \quad B_a = 0.$$

From 5.1 we derive

$$(4.5) \quad \frac{\partial}{\partial s} A_i - \frac{\partial}{\partial t} B_i = f_{jik}(S(s, t))B_jA_k.$$

To prove that B inherits a Riemannian metric from the given metric on X , it suffices to show that, for every such surface S ,

$$(4.6) \quad \frac{\partial}{\partial t} B_u B_u = 0$$

for then it would follow that the lengths of all horizontal liftings of a given tangent vector to B have the same length.

Using (4.4) and (4.5), we have

$$\frac{\partial}{\partial t} B_u B_u = 2B_u c_{auv} B_v A_a.$$

For each a , the matrix c_{avu} is the matrix of $\text{Ad } g_a$ acting on $\mathcal{M}$: by our assumption it is skew-symmetric, hence

$$B_u c_{avu} B_v = 0. \text{ This proves (4.6).}$$

We want to now show that the fibres of ϕ are totally geodesic. Let

then $\sigma: [0, 1] \rightarrow X$ be a curve in the fibre, i.e. $\sigma'(t)$ is vertical, and $v: t \rightarrow v(t) \in X_{\sigma(t)}$ a vertical vector field along σ , i.e. $w_u(v(t)) = 0$. From (4.3) we have

$$w_u(\Delta v) = -h_{uab}(\sigma(t))w_b(\sigma''(t))w_a(v(t)).$$

From 4.2

$$h_{uab} = \frac{1}{2} (c_{bau} - c_{aub} + c_{uba}).$$

That each of these three terms is zero follows from the relations $[K, K] \subset K$, $[K, M] \subset M$.

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