THE ISOLATED POINTS IN THE DUAL OF A COMMUTATIVE SEMI-GROUP[1]

W. W. COMFORT[2]

1. **Introduction.** It is well known that if the dual group $\hat{G}$ of a commutative group G is given the topology of point-wise convergence on G, then $\hat{G}$ admits an isolated point if and only if G is finite; in case G is finite, each point of $\hat{G}$ is isolated. The present paper offers an extension of this theorem to commutative semi-groups.

1.1. Definition. A commutative semi-group is a nonempty set G together with an operation (denoted by juxtaposition) on $G \times G$ to G for which $(ab)c = a(bc)$ and $ab = ba$ whenever $a, b, c \in G$.

1.2. Definition. A semi-character on the commutative semi-group G is a bounded, complex-valued, multiplicative function on G which is not identically zero. The space of semi-characters on G is denoted $\hat{G}$.

1.3. Definition. Let F be a finite subset of G, let $\epsilon > 0$, and let $\chi \in \hat{G}$. Then the (F, ϵ) neighborhood of χ is

$$U_{F,\epsilon}(\chi) = \{\psi \in \hat{G} \mid |\psi(z) - \chi(z)| < \epsilon \text{ for each } z \in F\}.$$

The topology on $\hat{G}$ is the smallest which makes each (F, ϵ) neighborhood of each $\chi \in \hat{G}$ open.

1.4. Remark. In §4 of [1], it is observed that if $G' = \{T_u \mid u \in G\}$, where $T_u = \{v \in G \mid \chi(u) = \chi(v) \text{ for all } \chi \in \hat{G}\}$, then the map $u \to T_u$ is a homomorphism of G onto G', multiplication in G' being given by $T_u T_v = T_{uv}$. Theorem 8 of [1] shows that $\hat{G}$ is isomorphically homeomorphic to $(G')^{\wedge}$ under the mapping $\chi \to \chi'$, where $\chi'(T_u) = \chi(u)$. We shall therefore restrict our attention to the case $G = G'$. That is, we shall suppose in what follows that $\hat{G}$ separates points in G, a condition (see 3.5 of [1]) equivalent to the semi-simplicity of the algebra $l_1(G)$.

2. **The Hewitt-Zuckerman decomposition of** G. The chief result of this paper, Theorem 5.2, depends heavily on the structural theorems for G given in [1]. In this section we give a definition and cite from [1] a number of results for later use.

2.1. Definition. For $u, v \in G$, write $u \sim v$ if, for each $\chi \in \hat{G}$, $\chi(u) = 0$ if and only if $\chi(v) = 0$. Let $H_u = \{v \in G \mid v \sim u\}$.

Received by the editors April 7, 1959.

[1] Certain of the results presented here appeared in the author's doctoral dissertation, University of Washington, June, 1958. Research on that dissertation was sponsored in part by the National Science Foundation under Grant G-2053.

[2] The author is indebted to Professor Edwin Hewitt for his encouragement and generous assistance during the preparation of this paper.

2.2. THEOREM. *$\sim$ is an equivalence relation on G. The mapping $u \rightarrow H_u$ is a homomorphism of G onto $\{H_u | u \in G\}$, multiplication in the latter space being given by $H_u H_v = H_{uv}$.*

2.3. THEOREM. *Let $u \in G$. Then H_u is a sub-semi-group of G. If $v \in H_u$ and $w \in H_u$ and $vu = wu$, then $v = w$.*

2.4. THEOREM. *For each $u \in G$,*

$$H_u = \{v \in G \mid |\chi(u)| = 1 \leftrightarrow |\chi(v)| = 1 \text{ for each } \chi \in \hat{G}\}.$$

2.5. THEOREM. *If $u \in G$ and H_u contains an idempotent, then H_u is a group.*

2.6. THEOREM. *If $u \in G$ and H_u is a nongroup, then there is a bounded, complex-valued, multiplicative function ψ on H_u for which $0 < |\psi(z)| < 1$ for each $z \in H_u$.*

2.7. THEOREM. *Under the multiplication given in 2.2, $\{H_u | u \in G\}$ is an idempotent semi-group. Hence it is a semi-lattice under the partial ordering $H_u \leqq H_v$ if and only if $H_u H_v = H_u$.*

3. A theorem on isolated characters.

3.1. DEFINITION. If $\chi \in \hat{G}$, then the support of χ, denoted $S(\chi)$, is the set $S(\chi) = \{z \in G | \chi(z) \neq 0\}$.

3.2. PROPOSITION. *For each $\chi \in \hat{G}$, $S(\chi)$ is a sub-semi-group of G, and $G \backslash S(\chi)$ is an ideal in G.*

3.3. NOTATION. Let $\Gamma = \{\chi \in \hat{G} |$ if $z \in G$, then $\chi(z) = 0$ or $|\chi(z)| = 1\}$.

3.4. DEFINITION.[3] A subset K of G is called a face of G if, for each u and v in G, $uv \in K$ if and only if $u \in K$ and $v \in K$.

3.5. PROPOSITION. *If $\chi \in \hat{G}$, then $S(\chi)$ is a face.*

3.6. DEFINITION. Let K be a face in G, L the union of all proper subfaces of K. The set $K \backslash L$ is denoted core K and is called the core of K.

3.7. DEFINITION. For each $z \in G$, let A_z be the intersection of all faces containing z.

3.8. PROPOSITION. *If K is a face and $z \in K$, then $H_z \subset K$.*

PROOF. The set A_z is a face whose characteristic function is a semi-character assuming the value 1 at z.

---

[3] The author wishes to thank the referee (a) in general, for his broad advice and counsel, and (b) specifically, for suggesting Definitions 3.4 and 3.6.

3.9. PROPOSITION. *For each* $z \in G$,

$$A_z = \{x \in G \mid H_x \geqq H_z\}$$
$$= \{x \in G \mid \textit{if } \chi \in \hat{G} \textit{ and } \chi(z) \neq 0, \textit{ then } \chi(x) \neq 0\}.$$

3.10. LEMMA. *Let H be an infinite commutative semi-group with cancellation, and let F be a finite subset of H. Then, for each $\epsilon > 0$, there is a point $\psi \in \hat{H}$ for which $|\psi| = 1$, $|\psi(z) - 1| < \epsilon$ for each $z \in F$, and ψ is not the function identically 1 on H.*

PROOF. CASE I: H is a group. Let T denote the subgroup of H generated by F, and suppose first that $T = H$. Then there are elements $z_1, z_2, \cdots, z_k$ in H for which $H = \{z_1\} \oplus \{z_2\} \oplus \cdots \oplus \{z_k\}$, where some z_j, say z_1, has infinite order. Each $z \in H$ has the form $z = \prod_{j=1}^{k} z_j^{r_j(z)}$, where each r_j is a homomorphism on H into either the integers or the integers modulo a certain integer m_j. Let $r = \max\{|r_1(z)| \mid z \in F\}$. Then $r > 0$. Let $\eta \in]0, 1[$ have the property that $|\exp(2\pi i\partial) - 1| < \epsilon$ whenever $|\partial| < \eta$, and let $\gamma \in]0, 1[$ have the property that $\gamma r < \eta$. Define $\psi(z) = \exp(2\pi i \gamma r_1(z))$.

If $T \neq H$, then we use the following theorem found, for example, in [2, p. 99]: if $\overline{T}$ is any commutative group with identity $\bar{e}$, and if $\bar{z} \in \overline{T}$ with $\bar{z} \neq \bar{e}$, then there is a character $\bar{\psi}$ on $\overline{T}$ for which $\bar{\psi}(\bar{z}) \neq \bar{\psi}(\bar{e}) = 1$. Taking $\overline{T} = H/T$ and $z \in H \setminus T$, we define $\psi(y) = \bar{\psi}(\bar{y})$ for each $y \in H$.

CASE II. H is a nongroup. We consider H to be a subset of the group $q(H)$ of equivalence classes of formal quotients of elements of H, the two quotients a_1/b_1 and a_2/b_2 being called equivalent if $a_1 b_2 = a_2 b_1$. From Case I we obtain a nontrivial character ψ on $q(H)$ for which $|\psi(z) - 1| < \epsilon$ whenever $z \in F$. The restriction of ψ to H is in $\hat{H}$, and if $a/b \in q(H) \setminus \psi^{-1}(1)$, then either $\psi(a) \neq 1$ or $\psi(b) \neq 1$.

3.11. THEOREM. *Let χ be isolated in $\hat{G}$. Then core $S(\chi)$ is a finite group.*

PROOF. We show first that core $S(\chi) \neq \Lambda$. Let the (F, ϵ) neighborhood of χ isolate χ in $\hat{G}$, and call this neighborhood U. We may clearly suppose that the set $F_1 = F \cap S(\chi)$ is nonempty. Let

$$F_1 = \{x_1, x_2, \cdots, x_n\},$$

and let $x = x_1 x_2 \cdots x_n$. Then $x \in S(\chi)$, so $A_x \subset S(\chi)$ by 3.9. If $S(\chi) \neq A_x$, then there is a point $y \in S(\chi)$ for which $H_y \ngeqq H_x$. Let ψ be that function on G for which $\psi(z) = \chi(z)$ if $z \in A_x$, $\psi(z) = 0$ if $z \notin A_x$. Then $\psi \in \hat{G}$ and ψ agrees with χ on F, so that $\psi \in U$. But $\psi(y) \neq 0$, so that $\psi \neq \chi$. It follows that $S(\chi) = A_x$, so that core $S(\chi) = H_x$.

Assume now that H_x is infinite. Then there is by 3.10 a point $\psi\in\hat{H}_x$ with $|\psi|=1$, ψ not identically 1, and $|\psi(z)-1|<\epsilon/2$ for each $z\in\{x, xx_1, xx_2, \cdots, xx_n\}$. We extend ψ to all of G by defining $\psi(z)=\psi(xz)/\psi(x)$ if $z\in A_x$, $\psi(z)=0$ if $z\notin A_x$. Then $\psi\in\hat{G}$ and, for each $x_i\in F_1$,

$$\begin{aligned} |(\chi\psi)(x_i)-\chi(x_i)| &\leqq |\psi(x_i)-1| \\ &= |\psi(xx_i)-\psi(x)| \\ &\leqq |\psi(xx_i)-1| + |\psi(x)-1| < \epsilon. \end{aligned}$$

We conclude that H_x is finite. Then by 2.5, H_x is a finite group.

3.12. Corollary. *Let χ be isolated in $\hat{G}$. Then $\chi\in\Gamma$.*

Proof. $|\chi|=1$ on the group core $S(\chi)$. Hence $|\chi(z)|=1$ for each $z\in S(\chi)$.

3.13. Definition. A character on G is a semi-character on G whose multiplicative inverse exists in $\hat{G}$.

3.14. Proposition. *Let $\chi\in\hat{G}$. Then χ is a character on G if and only if $S(\chi)=G$ and $\chi\in\Gamma$.*

3.15. Theorem. *Let $\chi\in\hat{G}$, with $S(\chi)=G$. Then, with $C=$ core G, the following statements are equivalent*:

(a) *C is a finite group*;
(b) *$C\neq\Lambda$ and 1 is isolated in $\hat{C}$*;
(c) *1 is isolated in $\hat{G}$*;
(d) *χ is isolated in $\hat{G}$.*

Proof. The implications (a)→(b) and (b)→(c) are obvious, and (d)→(a) is 3.8. Suppose (c) holds, and let the (F, ϵ) neighborhood of 1 isolate 1 in $\hat{G}$. Let U be the (F, ϵ) neighborhood of χ in $\hat{G}$. If $\psi\in U$, then $\chi\bar{\psi}=1$, so that $\psi=\chi$.

3.16. Corollary. *Let χ_1, $\chi_2\in\hat{G}$, with $S(\chi_i)=G$ for $i=1, 2$. Then χ_1 is isolated in $\hat{G}$ if and only if χ_2 is isolated in $\hat{G}$.*

4. **An extension theorem.** In this section we answer the following question: given a semi-character $\chi\in\Gamma$ with core $S(\chi)\neq\Lambda$ and a point $y\in G\backslash S(\chi)$, can a semi-character ψ be constructed which agrees with χ on $S(\chi)$ and which does not vanish at y? Theorem 4.2 shows that, unless the answer is obviously *no*, the answer is *yes*. The author does not know whether the hypothesis core $S(\chi)\neq\Lambda$ in Theorem 4.2 is essential.

4.1. Definition. Let $\chi\in\hat{G}$, and let $N(\chi)$, the negative set of χ, be all $y \in G \backslash S(\chi)$ for which there exist $x_1, x_2 \in S(\chi)$ with $\chi(x_1) \neq \chi(x_2)$ and $x_1 y = x_2 y$.

4.2. THEOREM. *Let $\chi \in \Gamma$, with core $S(\chi) = H_x$. Suppose that $y \in G \backslash S(\chi)$, and that $A_{xy} \cap N(\chi) = \Lambda$. Then there is a point $\psi \in \Gamma$ which agrees with χ on $S(\chi)$, which assumes at y a nonzero value, and which vanishes off of A_{xy}.*

PROOF. We may clearly suppose (replacing y by xy if necessary) that $H_y < H_x$. Well-order H_y. If $\psi(y_\lambda)$ has been defined for each $\lambda \in U \cup V$, and if

$$(*) \qquad x_1 \cdot \prod_{\lambda \in U} y_\lambda^{r_\lambda} = x_2 \cdot \prod_{\lambda \in V} y_\lambda^{r_\lambda}$$

for certain elements $x_1, x_2 \in H_x$ and certain integers $r_\lambda > 0$, and if

$$\chi(x_1) \cdot \prod_{\lambda \in U} [\psi(y_\lambda)]^{r_\lambda} = \chi(x_2) \cdot \prod_{\lambda \in V} [\psi(y_\lambda)]^{r_\lambda},$$

then we will say that χ and ψ multiply on the relation (*). For each index μ, $P(\mu)$ will be the statement that if $U \cup V \subset \{\lambda \mid \lambda \leqq \mu\}$, and if (*) holds, then χ and ψ multiply on (*).

Now let m be the smallest positive integer for which there are points x_1, x_2 in H_x for which $x_1 y_1 = x_2 y_1^{m+1}$ and, with k any complex number for which $k^m = \chi(x_1)\bar{\chi}(x_2)$, define $\psi(y_1) = k$. If no such m exists, let $\psi(y_1)$ be any complex number of absolute value unity. $P(1)$ follows from the cancellation law in H_y and a theorem of Euclid. If $\psi(y_\lambda)$ has been defined for each $\lambda < \mu$, and if $P(\lambda)$ holds for each $\lambda < \mu$, we let m be the smallest positive integer for which there is a relation of the form

$$y_\mu^m x_1 \cdot \prod_{\lambda \in U} y_\lambda^{r_\lambda} = x_2 \cdot \prod_{\lambda \in V} y_\lambda^{r_\lambda},$$

where $x_i \in H_x$, U and V are nonempty collections of indices preceding μ, and r_λ are positive integers. Then, with k any complex number for which $k^m = \chi(x_2)\bar{\chi}(x_1) \prod_{\lambda \in V} [\psi(y_\lambda]^{r_\lambda} \prod_{\lambda \in U} [\bar{\psi}(y_\lambda)]^{r_\lambda}$, we let $\psi(y_\mu) = k$. If no such m exists, we let $\psi(y_\mu)$ be any complex number of absolute value unity. Then $P(\mu)$ holds. The function ψ, when defined on all of H_y, is extended to all of G as follows: $\psi(z) = \psi(yz)/\psi(y)$ if $z \in A_y$, $\psi(z) = 0$ if $z \notin A_y$. Then ψ agrees with χ on H_x, hence on $S(\chi)$.

5. The isolated points of $\hat{G}$.

5.1. THEOREM. *If $\chi \in \Gamma$, then the following two statements are equivalent:*

(i) *there is a neighborhood* U *of* χ *in* $\hat{G}$ *for which* $U\cap\Gamma=\{\chi\}$.

(ii) *core* $S(\chi)$ *is a finite group and there is a finite subset* J *of* $G\backslash S(\chi)$ *for which* $A_{xy}\cap[J\cup N(\chi)]\neq\Lambda$ *whenever* $y\in G\backslash S(\chi)$ *and* $x\in$ *core* $S(\chi)$.

PROOF. (i)$\rightarrow$(ii). To see that core $S(\chi)$ is a finite group, copy the proof of 3.11, replacing the symbol $\hat{G}$ throughout by the symbol Γ.

Let U be the (F, ϵ) neighborhood of χ, and define $J=F\backslash S(\chi)$. If x and y are points of core $S(\chi)$ and $G\backslash S(\chi)$ respectively for which $A_{xy}\cap[J\cup N(\chi)]=\Lambda$, then the function ψ given by 4.2 agrees with χ on F, differs from χ at y, and is in Γ.

(ii)$\rightarrow$(i). Let $\epsilon=1/4$ if $n=1$, $\epsilon=|1-e^{2\pi i/n}|/2$ if $n>1$, where $n=$card core $S(\chi)$. Let $F=$core $S(\chi)\cup J$. Then the (F, ϵ) neighborhood of χ is as required in (i).

5.2. THEOREM. *If* $\chi\in\hat{G}$, *then the following two statements are equivalent*:

(i) χ *is an isolated point in* $\hat{G}$;

(ii) *core* $S(\chi)$ *is a finite group and there is a finite set* I *of idempotents in* $G\backslash S(\chi)$ *for which* $A_{xy}\cap[I\cup N(\chi)]\neq\Lambda$ *whenever* $y\in G\backslash S(\chi)$ *and* $x\in$*core* $S(\chi)$.

PROOF. (i)$\rightarrow$(ii). The first part of (ii) is 3.11. Let the (F, ϵ) neighborhood isolate χ, and let core $S(\chi)=H_x$ have n elements. For each $u\in F\backslash S(\chi)$ for which $A_{ux}\backslash A_x$ contains at least one idempotent, pick one such idempotent, e_u. Let I be the collection of idempotents chosen, so that card $I\leqq$card F. If (ii) fails, then there is by 4.2 a $y\in G\backslash S(\chi)$ and $\psi\in\Gamma$ for which $S(\psi)=A_{xy}$, ψ agrees with χ on $S(\chi)$, and ψ vanishes on I. We may suppose (since $[S(\psi)\cap F]\backslash S(\chi)\neq\Lambda$) that $y\in F$. Hence we may suppose that H_{xy} is covered by H_x, in the sense that there is no $z\in G$ for which $H_{xy}<H_z<H_y$. Let $\{y_1, y_2, \cdots, y_k\}=[A_{xy}\cap F]\backslash S(\chi)$, and let e be the identity of the group H_x. Then $ey_i\in H_{xy}$ for $1\leqq i\leqq k$. There is by 2.6 a multiplicative function ω on H_{xy} for which $0<|\omega(z)|<1$ whenever $z\in H_{xy}$. We may suppose, replacing ω by one of its integral powers if necessary, that $|\omega(ey_i)|<\epsilon$ for $1\leqq i\leqq k$. Extend ω to all of G as follows: if $z\in A_{xy}$, then $\omega(z)=\omega(xyz)/\omega(xy)$; if $z\notin A_{xy}$, then $\omega(z)=0$. Then ω is multiplicative. $\omega(e)=1$ so $|\omega|=1$ on H_x, hence on A_x. If $z\in A_{xy}\backslash A_x$, then $ez\in H_{xy}$, so that $|\omega(z)|=|\omega(ez)|<1$. Hence $\omega\in\hat{G}$. We obtain the desired contradiction by noting that ω^n is identically 1 on A_x, so that $\omega^n\psi\in U$.

(ii)$\rightarrow$(i). First rewrite the proof "(ii)$\rightarrow$(i)" of 5.1 with J replaced by I. The resulting neighborhood U of χ has the property that $U\cap\Gamma=\{\chi\}$. If $\psi\in U\cap\hat{G}$, then ψ agrees with χ on $S(\chi)$ and ψ vanishes on I. Defining $\omega(z)=\psi(z)/|\psi(z)|$ if $z\in S(\psi)$, $\omega(z)=0$ otherwise, we have $\omega\in U\cap\Gamma$. Hence $\omega=\chi$, so that $S(\psi)=S(\omega)=S(\chi)$ and $\psi=\chi$.

Bibliography

1. Edwin Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. vol. 83 (1956) pp. 70–97.

2. André Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Sci. Ind., no. 869, Paris, Hermann, 1940.

University of Washington and
Harvard University

HAUSDORFF INTERVAL TOPOLOGY ON A PARTIALLY ORDERED SET

YATARŌ MATSUSHIMA

We shall generalize a condition of E. S. Wolk [1] that the interval topology in a partially ordered set be Hausdorff. Let X be a partially ordered set. For each $a \in X$, let $N(a)$ be the set of all elements of X, noncomparable with a. We introduce the definition of an "*a-separating set*": any subset S of $N(a)$ such that every $x \in N(a)$ is comparable with some $y \in S$.

Theorem. *If each $a \in X$ has a finite a-separating set, then X is a Hausdorff space in its interval topology.*

Proof. Let $a \neq b$, $a, b \in X$. Let $\{a_i\}_1^r(\{b_j\}_1^s)$ be an *a-separating* (*b-separating*) *set*; we define for each of the cases the sets A and B; one checks easily in each case that A, B have the stated properties.

(1) The case where a, b are comparable.

(α) Let $a<b$. If there is an element c such that $a<c<b$, then, there exists a *c-separating set* $\{c_i\}$, so that

$$N(c) \subset \sum_{i=1}^{m} ([-\infty, c_i] + [c_i, \infty]), \quad \text{where} \quad c_i \in N(c).$$

In this case if we put

$$A = [-\infty, c] + \sum_{i=1}^{m} [-\infty, c_i], \qquad B = [c, \infty] + \sum_{i=1}^{m} [c_i, \infty],$$

Received by the editors April 17, 1959.