ON "AN IDENTITY IN ARITHMETIC"[1]

GERARD DEBREU

H. D. Block and J. Marschak have presented in the Bulletin of the American Mathematical Society vol. 65 (1959) pp. 123–124, an identity which arose in a probability context. This note proves it by a probability theoretical argument.

Consider an experiment having the set of possible outcomes $N=\{1, \cdots, n\}$ with positive probabilities $(u_1, \cdots, u_n)$ and an infinite sequence of independent repetitions of that experiment. Let M be a subset of N and i be an element of M, and denote by $A(i, M)$ the event that i is the first element of M which occurs in an infinite sequence of outcomes. If $B_j(i, M)$ denotes the event that the first $(j-1)$ outcomes are not in M *and* the jth outcome is i, one has

$$A(i, M) = \bigcup_{j=1}^{\infty} B_j(i, M).$$

Hence

$$\Pr[A(i, M)] = \sum_{j=1}^{\infty} \Pr[B_j(i, M)] = \sum_{j=1}^{\infty} (1 - u_M)^{j-1} u_i = u_i/u_M,$$

writing u_M for $\sum_{j\in M} u_j$, and putting $0^0=1$ for the degenerate case $M=N$.

Let now r be a permutation of N and k_r be the element of N ranked kth by r. Let also $C(r)$ be the event that the first occurrences of the n elements of N in an infinite sequence of outcomes appear in the order $r=(1_r, \cdots, n_r)$. If $D_{j_1,\cdots,j_n}(r)$ denotes the event that k_r occurs for the first time at the j_kth performance of the experiment for every $k=1, \cdots, n$ (therefore $j_1=1$ in what follows), one has

$$C(r) = \bigcup_{j_1<\cdots<j_n} D_{j_1,\cdots,j_n}(r).$$

Hence

$$\Pr[C(r)] = \sum_{j_1<\cdots<j_n} \Pr[D_{j_1,\cdots,j_n}(r)]$$

$$= \sum_{j_1<\cdots<j_n} u_{1_r}^{j_2-j_1} u_{2_r} (u_{1_r} + u_{2_r})^{j_3-j_2-1} u_{3_r} (u_{1_r} + u_{2_r} + u_{3_r})^{j_4-j_3-1} \cdots u_{n_r}.$$

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Putting $j_{k+1}-j_k-1=h_k$ and $u_{1_r}+\cdots+u_{k_r}=v_{k,r}$, one obtains

$$\Pr[C(r)] = \left(\prod_{j=1}^{n} u_j\right) \sum_{h_1,\cdots,h_{n-1}\geq 0} v_{1,r}^{h_1} v_{2,r}^{h_2} \cdots v_{n-1,r}^{h_{n-1}}$$
$$= \left(\prod_{j=1}^{n} u_j\right) \prod_{k=1}^{n-1} (1 - v_{k,r})^{-1}.$$

Let finally $R(i, M)$ be the set of permutations of N for which i is ranked first among the elements of M. Since the event that some element of N never occurs has probability zero, one has

$$\Pr[A(i, M)] = \sum_{r\in R(i,M)} \Pr[C(r)].$$

Hence by a simple transformation we obtain the desired identity:

$$u_i \Big/ \sum_{j\in M} u_j = \left(\prod_{j=1}^{n} u_j\right) \sum_{r\in R(i,M)} \prod_{k=1}^{n} \left(\sum_{j=k}^{n} u_{j_r}\right)^{-1}.$$

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