REMARKS ON CAUCHY'S INTEGRAL FORMULA IN MATRIC SPACES

JOSEPHINE MITCHELL

1. Introduction. Recently several proofs of Cauchy's integral formula have been given for matric spaces [**2**; **4**; **5**; **7**]. However a short direct proof is available by using the argument that Morita gives to prove the Poisson formula (see §2). In §3 the formula is also proved by means of a minimal problem, similar to those introduced by Bergman [**1**]. Since the present paper is closely related to Morita's [**7**], we use his notation wherever possible.

The matric spaces under consideration are the four main types of irreducible bounded symmetric domains given by E. Cartan [**3**]. Let z be a matrix of complex numbers, z' its transpose, z^* its conjugate transpose and $I^{(r)}$ the identity matrix of order r. Then the first three types are defined by

$$D = E[z \mid I^{(n)} - z^*z > 0], \tag{1}$$

where

I. $\mathfrak{A}_{mn}$: z is a matrix of type $(m, n)(m \geqq n)$.

II. $\mathfrak{S}_n$: z is a symmetric matrix of order n.

III. $\mathfrak{K}_n$: z is a skew symmetric matrix of order n.

The fourth type is

IV. $\mathfrak{M}_n$: the set of all matrices z of type $(n, 1)$ (that is, n-dimensional vectors) such that

$$|z'z| < 1, \qquad 1 - 2z^*z + |z'z|^2 > 0. \tag{2}$$

It is known that each of the domains possesses a distinguished boundary B [**1**], which is defined by

$$z^*z = I^{(n)} \tag{3}$$

for $\mathfrak{A}_{mn}$, $\mathfrak{S}_n$ and for $\mathfrak{K}_n$ if n is even, or the eigenvalues of z^*z are all 1 except one which is zero if n is odd. For $\mathfrak{M}_n$, B is given by

$$z^*z = 1, \qquad |z'z| = 1. \tag{4}$$

2. Cauchy's integral formula. We define a kernel function (the Cauchy kernel) by

$$K(z, \zeta) = V^{-1} \det{}^{-p} (z - \zeta), \tag{5}$$

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where $p=n$ for domains $\mathfrak{A}_{nn}$, $(n+1)/2$ for $\mathfrak{S}_n$, $(n-1)/2$ for $\mathfrak{L}_n$ if n is even, and V is the Euclidean volume of the domain B. For domains $\mathfrak{M}_n$

$$K(z, \zeta) = V^{-1}[(z-\zeta)'(z-\zeta)]^{-n/2}. \tag{6}$$

Then

THEOREM 1. *Let $f(z)$ be regular in D and continuous on $\overline{D}$ (the closure of D), where D is one of the domains $\mathfrak{A}_{nn}$, $\mathfrak{S}_n$, $\mathfrak{L}_{2n}$ or $\mathfrak{M}_n$. Then*

$$f(\zeta) = \int_B K(\zeta, z) f(z)\dot{z}, \qquad \zeta \in D, \tag{7}$$

where

$$\begin{aligned} \dot{z} &= c_{n1} \prod_{j,k=1}^{n} dz_{jk} && \text{for } \mathfrak{A}_{nn} \\ &= c_{n2} \prod_{j=1;k\geq j}^{n} dz_{jk} && \text{for } \mathfrak{S}_n \\ &= c_{n3} \prod_{j=1;k>j}^{n-1} dz_{jk} && \text{for } \mathfrak{L}_n \ (n \text{ even}) \\ &= c_{n4} \prod_{j=1}^{n} dz_j && \text{for } \mathfrak{M}_n, \end{aligned} \tag{8}$$

and the constants c_{nj} are such that $V^{-1}\int_B K^{-1}(z, 0)\dot{z} = 1$ in each case. (We note that $K^{-1}(z, 0)\dot{z}$ is the Euclidean volume element for the set B [7].)

PROOF. We shall restrict ourselves to the case $D = \mathfrak{A}_{nn}$ but the other cases may be treated similarly. See [7, §16] for the details in the case $\mathfrak{M}_n$.

It is known that the set of analytic mappings taking D onto itself and ζ into 0 and the inverse transformations are given by [8]

$$\begin{aligned} w &= a(z-\zeta)(d - d\zeta^* z)^{-1}, \\ z &= \sigma(w) = (a + wd\zeta^*)^{-1}(wd + a\zeta) \\ &\qquad\quad = (a^*w + \zeta d^*)(d^* + \zeta^* a^* w)^{-1}, \end{aligned} \tag{9}$$

subject to the conditions

$$\begin{aligned} &a(I - \zeta\zeta^*)a^* = I, \qquad d(I - \zeta^*\zeta)d^* = I, \qquad \zeta^* a^* a = d^* d \zeta^*, \\ &a^*a - \zeta d^* d \zeta^* = I, \qquad d^*d - \zeta^* a^* a \zeta = I, \end{aligned} \tag{9a}$$

$(I = I^{(n)})$. Also these transformations leave the set B invariant.

Suppose first that $f(z)$ is analytic on $\bar{D}$ and consider the expression

$$F(\zeta, z, z^*, \dot{z}) = f(z) \det{}^{-n} (I - z^*\zeta) dv_z,$$

where dv_z is the Euclidean volume element of the set B:

$$dv_z = \det{}^{-n} z\dot{z}.$$

Under the transformation (9)

$$\dot{z} = \frac{\partial(z)}{\partial(w)} \dot{w}$$

but [6]

$$\frac{\partial(z)}{\partial(w)} = \det{}^{-n} (a + wd\zeta^*) \det{}^{-n} (d^* + \zeta^* a^* w).$$

Also since $w^* w = I$,

$$dv_z = \det{}^{-n} (d + w^* a\zeta) \det{}^{-n} (d^* + \zeta^* a^* w) dv_w,$$

and

$$I - z^*\zeta = (d + w^* a\zeta)^{-1} d(I - \zeta^*\zeta).$$

Thus

$$\int_B F(\zeta, z, z^*, \dot{z}) = \det{}^{-n} d \det{}^{-n} (I - \zeta^*\zeta) \int_B f_0(w) dv_w.$$

where

$$f_0(w) = \det{}^{-n} (d^* + \zeta^* a^* w) f(\sigma(w)) \tag{10}$$

is regular on $\bar{D}$.

By a theorem due to H. Cartan a regular function on $\bar{D}$ can be expanded on $\bar{D}$ into a uniformly convergent series of homogeneous polynomials, $\sum_{n=0}^{\infty} a_n P_n(w)$, where $P_0(w)$ is a constant so that $a_0 P_0 = f_0(0)$ and P_n is of degree >0 for $n>0$. Also since B is circular [5]

$$\int_B P_n(w) dv_w = 0 \qquad \text{for } n > 0.$$

Thus

$$\int_B f_0(w) dv_w = V f_0(0)$$

and by (9a) and (10)

$$(1/V)\int_B F(\zeta, z, z^*, \dot{z}) = f(\zeta).$$

Since on $z^*z = I$,

$$\det{}^{-n}(I - z^*\zeta)dv_z = \det{}^{-n}(z - \zeta)\dot{z},$$

(7) follows for functions regular on $\bar{D}$.

In case $f(z)$ is regular on D and continuous on $\bar{D}$, following Morita, we have

$$f(t\zeta) = \int_B K(\zeta, z)f(tz)dv_z$$

for any real number t such that $0 \leqq t < 1$. Letting $t \to 1^-$ we see that (7) holds for such a function $f(z)$. Thus Theorem 1 is proved.

3. **Minimal problem.** Let D be an arbitrary bounded domain in the space of n complex variables with a distinguished boundary B. Let ζ be an arbitrary fixed point of D and consider the subclass S of regular functions f on $\bar{D}$ such that $f(\zeta) = 1$. Suppose there exists a function $M(\zeta, z)$ of S which minimizes the integral

$$\int_B |f(z)|^2 dv_z, \qquad \in S.$$

Then defining

$$K_0^*(\zeta, z) = M(\zeta, z) \Big/ \int_B M(\zeta, w)dv_w, \tag{11}$$

we have

$$\frac{K_0^*(\zeta, z)}{K_0^*(\zeta, \zeta)} = \frac{M(\zeta, z)}{\int_B M(\zeta, w)dv_w} \cdot \int_B \frac{M(\zeta, w)dv_w}{M(\zeta, \zeta)} = M(\zeta, z).$$

In analogy to the case of one complex variable we call the function $K_0(\zeta, z)$ the *Szegö kernel* of the domain D.

Theorem 2. *Let $M(\zeta, z)$ be a solution of the above minimal problem and $K_0(\zeta, z)$ the kernel defined by* (11). *Then for any f regular on $\bar{D}$ the* (*reproducing*) *formula*

$$f(\zeta) = \int_B K_0(\zeta, z)f(z)dv_z, \qquad \zeta \in D. \tag{12}$$

holds. Also the minimum value of the integral is $[1/K_0(\zeta, \zeta)]$.

PROOF. From the minimal property for any arbitrary complex ϵ and regular f

$$\int_B |M|^2 dv \leqq \int_B |M + \epsilon[f(z) - f(\zeta)]|^2 dv.$$

Thus

$$2 \operatorname{Re}\left[\epsilon \int_B M^*[f(z) - f(\zeta)]dv\right] + |\epsilon|^2 \int_B |f(z) - f(\zeta)|^2 dv \geqq 0.$$

Since $|\epsilon|$ and $\arg \epsilon$ are both arbitrary, it follows that

$$\int_B M^*(\zeta, z)[f(z) - f(\zeta)]dv = 0,$$

from which (12) results. Also from (12)

$$\int_B |M(\zeta, z)|^2 dv = |K_0^{-2}(\zeta, \zeta)| \int_B K_0(\zeta, z) K_0^*(\zeta, z) dv$$
$$= K_0^{-1}(\zeta, \zeta).$$

For the matric spaces as we have seen in §2 this formula is valid for any f regular on D and continuous on $\overline{D}$.

For the domains $\mathfrak{A}_{nn}$, $\mathfrak{S}_n$ and $\mathfrak{L}_{2m}$ the kernel $K_0(\zeta, z)$ is equal to

$$K_0(\zeta, z) = V^{-1} \det{}^{-p} (I - z^*\zeta), \tag{13}$$

which equals $\det^p z\, K(\zeta, z)$ if $z \in B$, where p has the same values as for the kernel $K(\zeta, z)$; for $\mathfrak{M}_n$

$$K_0(\xi, z) = V^{-1}[1 - 2z^*\zeta + (\zeta'\zeta)(z'z)^*]^{-n/2}. \tag{13a}$$

The proof that the minimal problem has a solution for the matric spaces and that (13) satisfies (11) is similar to that in [6] for the Bergman kernel function and will be omitted here.

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The Pennsylvania State University

A COUNTABLE INTERPOLATION PROBLEM

Z. A. MELZAK

1. Let $\mathfrak{H}$ be the set of all order-preserving homeomorphisms of $I=[0, 1]$ onto itself. $\mathfrak{H}$ is a metric space in the uniform metric ρ:

$$\rho(f_1, f_2) = \max_I |f_1(x) - f_2(x)|, \qquad f_1, f_2 \in \mathfrak{H}. \tag{1}$$

Franklin [1] has proved the following theorem: (A) Let A and B be two countable sets, each dense on I. Then the set of analytic $f\in\mathfrak{H}$, such that $f(A)=B$, is dense in $\mathfrak{H}$.

It follows from (A) and from its extension in [2] that there exist nontrivial analytic functions $f\in\mathfrak{H}$, such that $f(x)$ is transcendental for each transcendental $x\in I$, and for each algebraic $x\in I$, x and $f(x)$ are algebraic and of the same degree.

In this note, without using either of these results, we prove a similar but complementary statement by means of Baire's Category Theorem.

THEOREM 1. *Let $\mathfrak{H}_\alpha$, $\alpha>2$, be the subset of $\mathfrak{H}$ consisting of all functions $f\in\mathfrak{H}$, whose values are either rational or transcendental and approximable to degree $>\alpha$, for each algebraic $x\in I$. Then $\mathfrak{H}_\alpha$ is a dense G_δ-set of second category in $\mathfrak{H}$.*

2. Since $\mathfrak{H}$ is not complete in ρ, we first remetrize it. Let

$$\sigma(f_1, f_2) = \rho(f_1, f_2) + \rho(f_1^{-1}, f_2^{-1}), \qquad f_1, f_2 \in \mathfrak{H}. \tag{2}$$

LEMMA 1. *$\mathfrak{H}$ is complete in the σ-metric.*

Let $\mathfrak{F}=I^I$ be the set of all continuous maps from I into I, then $\mathfrak{F}$ is complete in ρ. Let $\{f_n\}$, $n=1, 2, \cdots$, be a σ-Cauchy sequence in $\mathfrak{H}$. Then $\{f_n\}$ is also a ρ-Cauchy sequence in $\mathfrak{F}$, therefore $f_n\to f$,

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