A THEOREM ON GENERAL RECURSIVE FUNCTIONS

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The purpose of this note is to prove the following theorem.

THEOREM. *Every general recursive function $f(n)$ is expressible in the form $f(n) = g(2^n)$ for some function g defined by the following schema*

$$g(0) = 0,$$
$$g(n') = \theta(n', g(\delta(n')))$$

where θ and δ are primitive recursive and $\delta(n') \prec n'$ in some primitive recursive well-ordering of the natural numbers of order type ω.

This theorem is an improved version of the result obtained by Myhill [**1**] as well as a similar one obtained by Routledge [**2**, Theorem 4]. As functions of one variable are concerned, the foregoing theorem is the same as Routledge's except that the well-ordering $\prec$ referred to in this note is primitive recursive while it is defined by general recursive processes in Routledge's paper (see [**2**, Theorem 2]). Myhill's statement is, however, not sufficiently explicit. A more explicit statement of his result was supplied by Kleene in a letter to the author. The theorem in this note differs from Kleene's formulation only in that Kleene expressed $f(n)$ as $U(g(2^n))$ while here the function U is not used. This leads to the result $f(n) = g(2^n)$, which is, according to Routledge (see [**2**, §§8 and 9]), the best possible in the sense that we can no longer omit the function 2^n and therefore can not simply equate $f(n)$ and $g(n)$.

PROOF OF THE THEOREM. Let $f(n)$ be any given general recursive function. According to a theorem by Kleene [**3**, p. 288] there are two primitive recursive functions $K(n)$ and $T(x, y)$ such that

$$(1) \qquad (x)(Ey)(T(x, y) = 0) \quad \text{and} \quad f(n) = K(\mu y(T(n, y) = 0))$$

where $\mu y(T(n, y) = 0)$ is the least y such that $T(n, y) = 0$.

In terms of $K(n)$ and $T(x, y)$ we define the function $g(n)$ in the following manner:

$$g(0) = 0,$$
$$g(n') = \begin{cases} g(2^x \cdot 3^{y+1}), \text{ if } n' = 2^x \cdot 3^y \text{ and } T(x, i) \neq 0 \text{ for } i < y + 1, \\ K(y), \text{ if } n' = 2^x \cdot 3^y \text{ and } T(x, y) = 0 \text{ but } T(x, i) \neq 0, \text{ if } i < y, \\ 0, \text{ otherwise.} \end{cases}$$

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Consequently we have, for every n,

$$g(2^n) = g(2^n \cdot 3^1) = g(2^n \cdot 3^2) = \cdots = g(2^n \cdot 3^{q_n - 1}) = g(2^n \cdot 3^{q_n}) = K(q_n)$$

where $q_n = \mu y(T(n, y) = 0)$. By the second expression of (1) we then have

$$f(n) = g(2^n), \tag{2}$$

In order to reformulate this definition of g we first introduce the following notations:

$A(n) = 0 \leftrightarrow n$ can be written in the form $2^x \cdot 3^y$ and $T(x, i) \neq 0$ for $i < y+1$.

$B(n) = 0 \leftrightarrow n$ can be written in the form $2^x \cdot 3^y$ and $T(x, y) = 0$ and $T(x, i) \neq 0$, if $i < y$.

Following the notations in Kleene's book [3] we can show that[1]

$$A(n) = 0 \leftrightarrow n = 2^{(n)_0} \cdot 3^{(n)_1} \ \& \prod_{i<(n)_1+1} T((n)_0, i) \neq 0,$$

$$B(n) = 0 \leftrightarrow n = 2^{(n)_0} \cdot 3^{(n)_1} \ \& \ T((n)_0, (n)_1) = 0,$$
$$\& \prod_{i<(n)_1} T((n)_0, i) \neq 0.$$

We see that $A(n)$ and $B(n)$ are primitive recursive functions. In terms of these and some other primitive recursive functions[2] we can reformulate the definition of $g(n)$ as follows:

$$g(0) = 0,$$
$$g(n') = \bar{sg}(A(n')) \cdot g(\bar{sg}(A(n')) \cdot 2^{(n')_0} \cdot 3^{(n')_1+1}) + \bar{sg}(B(n')) \cdot K((n')_1).$$

Let $\theta(n', x)$ stand for $\bar{sg}(A(n')) \cdot x + \bar{sg}(B(n')) \cdot K((n')_1)$ and $\delta(n')$ stand for $\bar{sg}(A(n')) \cdot 2^{(n')_0} \cdot 3^{(n')_1+1}$. Then the definition of g can be written as

$$g(0) = 0,$$
$$g(n') = \theta(n', g(\delta(n')))$$

where $\theta(n, x)$ and $\delta(n)$ are both primitive recursive.

To complete the proof it remains only to find a primitive recursive well-ordering $\prec$ of the natural numbers of type ω and then show that $\delta(n') \prec n'$ for every n. We first introduce the notations[3] $x \in C$ and $H(x)$

[1] We note, in particular, that $(0)_i = 0$, $(2^x \cdot 3^y)_0 = x$ and $(2^x \cdot 3^y)_1 = y$; for any function ψ, $\prod_{i<z} \psi(i) = 1$, if $z = 0$.

[2] $\bar{sg}(n) = 0$, if $n \neq 0$; $\bar{sg}(n) = 1$, if $n = 0$.

[3] See [3, p. 225]. For any predicate $R(x)$, $\mu t_{t<z} R(t)$ is the least $t < z$ such that $R(t)$, if $(Et)_{t<z} R(t)$; otherwise, z.

defined as follows:

$$x \in C \leftrightarrow A(x) = 0 \vee B(x) = 0,$$
$$H(x) = x \dot{-} \mu t_{t<x}((x \dot{-} t) \in C).$$

$H(x)$ is primitive recursive and has the following properties:

1. $H(x) = x$, if and only if $x=0$ or $x \in C$;
2. $H(x) \leqq x$ and $H(x') \in C$;
3. $H(x')$ is the number y nearest to x' such that $y \leqq x'$ and $y \in C$.

In terms of $H(x)$ a primitive recursive binary relation $\prec$ is defined by

$$x \prec y \leftrightarrow (H(x) = H(y) \,\&\, x < y) \vee ((H(x))_0 < (H(y))_0) \vee ((H(x))_0 = (H(y))_0 \,\&\, H(x) > H(y)).$$

According to this order $\prec$ all the natural numbers are arranged in a linear order as follows:

$$0 \prec 2^0 \cdot 3^{q_0} \prec \cdots \prec 2^0 \cdot 3^0 \prec \cdots \prec 2^1 \cdot 3^{q_1} \prec \cdots \prec 2^1 \cdot 3^0 \prec \cdots$$
$$\prec 2^i \cdot 3^j \prec 2^i \cdot 3^j + 1 \prec 2^i \cdot 3^j + 2 \prec \cdots \prec 2^i \cdot 3^j + \sigma(i, j) \prec 2^u \cdot 3^v$$
$$\prec \cdots$$

where $2^i \cdot 3^j \in C$; $q_n = \mu y(T(n, y) = 0)$; $u = i$ and $v = j-1$, if $j \neq 0$; but $u = i+1$ and $v = q_{i+1}$, if $j = 0$; $2^i \cdot 3^j + \sigma(i, j)$ is the greatest number y such that $H(y) = 2^i \cdot 3^j$. The value of $2^i \cdot 3^j + \sigma(i, j)$ is a finite number and can be calculated in the following manner. Let $2^i \cdot 3^j + 1 = k$. Then $2^i \cdot 3^j < k < 2^k$ and $2^k = 2^k \cdot 3^0 \in C$. It must be that $H(z) \neq 2^i \cdot 3^j$ for every $z > 2^k$. For, otherwise, there would be a $z > 2^k$ such that $H(z) = 2^i \cdot 3^j$. Then according to the third property of $H(x)$, $2^i \cdot 3^j$ would be the number y nearest to z such that $y \leqq z$ and $y \in C$. But this contradicts the fact that $2^k \in C$ and $2^i \cdot 3^j < 2^k < z$. Thus we have

$$2^i \cdot 3^j + \sigma(i, j) = 2^k \dot{-} \mu t_{t<2^k}(H(2^k \dot{-} t) = 2^i \cdot 3^j). \tag{3}$$

The above ordering shows that $x \prec y$ is a primitive recursive well-ordering of the natural numbers of type ω with 0 as the first element.

Now let us consider the function

$$\delta(n') = \bar{s}\bar{g}(A(n')) \cdot 2^{(n')_0} \cdot 3^{(n')_1 + 1}$$

which appears in the third definition of g. In case $A(n') = 0$ we have $n' = 2^{(n')_0} \cdot 3^{(n')_1} \in C$. Then $\bar{s}\bar{g}(A(n')) = 1$ and $\delta(n') = 2^{(n')_0} \cdot 3^{(n')_1+1}$. $\delta(n')$ must belong to C. By the first property of $H(x)$, we have $H(n') = n' = 2^{(n')_0} \cdot 3^{(n')_1}$ and $H(\delta(n')) = \delta(n') = 2^{(n')_0} \cdot 3^{(n')_1+1}$. Consequently,

$$(H(\delta(n')))_0 = (H(n'))_0 = (n')_0 \,\&\, H(\delta(n')) > H(n').$$

According to the definition of $\prec$ we have $\delta(n') \prec n'$. In case $A(n') \neq 0$ we have $\bar{s}\bar{g}(A(n'))=0$. Then $\delta(n')=0$. Since 0 is the first element in the ordering $\prec$ we also have $\delta(n') \prec n'$. Thus in any case $\delta(n') \prec n'$. This shows that the definition of $g(n)$ is an ordinal recursion schema with respect to $\prec$. This completes the proof of the theorem.

References

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