A REMARK CONCERNING THE AREA OF A NONPARAMETRIC SURFACE

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1. Let $\mathcal{O}$ be the set of quasi-linear real functions on the square $I = [0, 1] \times [0, 1]$; i.e., $p \in \mathcal{O}$ if p is continuous and there is a triangulation of I , on each triangle of which p is linear. The graph of p is composed of triangles, and the sum of their areas is called the elementary area, $E(p)$, of the surface determined by p . If we let

$$d(p, q) = \max[|p(x, y) - q(x, y)| : (x, y) \in I],$$

then $\mathcal{O}$ becomes a metric space, and E is a lower semi-continuous functional on $\mathcal{O}$. By a general theorem of Fréchet, E may be extended to a lower semi-continuous functional A on the completion $\mathcal{C} = \mathcal{O}'$ of $\mathcal{O}$. For every $f \in \mathcal{C}$,

$$A(f) = \inf \left[\liminf_{n \rightarrow \infty} E(p_n) : \lim_{n \rightarrow \infty} d(f, p_n) = 0 \right].$$

$\mathcal{C}$ is the space of continuous functions on I , $A(f)$ is known as the Lebesgue area of f , and if B is any lower semi-continuous extension of E from $\mathcal{O}$ to $\mathcal{C}$, then $B(f) \leq A(f)$, for every $f \in \mathcal{C}$.

In a previous paper, [1],² we considered another metric for $\mathcal{O}$; namely,

$$\delta(p, q) = \int_0^1 \int_0^1 |p(x, y) - q(x, y)| \, dx dy.$$

We showed that E is lower semi-continuous on $\mathcal{O}$, metrized by δ , so that, by Fréchet's theorem, E may be extended to a lower semi-continuous α on the space of equivalence classes of summable functions. We then showed that for every $f \in \mathcal{C}$, $A(f) = \alpha(f)$, so that we have a "legitimate" extension of Lebesgue area to all summable functions.

2. In this note, we are concerned with the possible metrics for $\mathcal{O}$, and wish to know:

(a) Under what conditions on a metric Δ is E lower semi-continuous on $\mathcal{O}$?

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² Numbers in brackets refer to the references at the end of the paper.

Assuming that (a) holds,

(b) Under what conditions on Δ does the resulting Fréchet extension agree with the Lebesgue area for continuous f ?

Condition (b) contains an implicit demand that the topology given by Δ be coarser than the topology of uniform convergence. On the other hand, a natural demand of analysis is that the topology given by Δ be finer than that given by convergence in measure. It is interesting that any metric, delimited in the above way, is adequate for our purpose, as the following theorem shows.

THEOREM. *If Δ is a metric for $\mathcal{P}$, such that the topology given by Δ is coarser than the topology of uniform convergence but finer than the topology of convergence in measure, then E is lower semi-continuous on $\mathcal{P}$, metrized by Δ . Moreover, if α_Δ is the lower semi-continuous Fréchet extension of E to the completion $\mathcal{P}_\Delta'$ of $\mathcal{P}$, then $\alpha_\Delta(f) = A(f)$ for every continuous f .*

3. In this section, we present some classical facts on nonparametric area [2], in a form adapted to our needs.

Let f be continuous on I . For every oriented rectangle $R = [a, b] \times [c, d]$ in I , let

$$G(f, R) = [\{G_1(f, R)\}^2 + \{G_2(f, R)\}^2 + |R|^2]^{1/2},$$

where

$$G_1(f, R) = \int_c^d |f(b, y) - f(a, y)| dy$$

and

$$G_2(f, R) = \int_a^b |f(x, d) - f(x, c)| dx.$$

Then, let

$$H(f) = \sup \left[\sum_{i=1}^n G(f, R_i) \right],$$

where $R_1, \dots, R_n$ are pair-wise disjoint oriented rectangles in I . In [2], H is defined in a slightly different way, but the proof that, for every $f \in \mathcal{C}$, $H(f) = A(f)$, follows the same lines for both forms of the definition.

In [3], a variant of H was introduced. Let $M_1(f, R)$ and $M_2(f, R)$ be the measures of the projections in the yz and xz planes, respectively, of the graph of the restriction of f to R , and let

$$M(f, R) = [\{M_1(f, R)\}^2 + \{M_2(f, R)\}^2 + |R|^2]^{1/2}.$$

Then let

$$K(f) = \sup \left[\sum_{i=1}^n M(f, R_i) \right],$$

where $R_1, \dots, R_n$ are pairwise disjoint oriented rectangles in I . The following hold. For every R , $M(f, R) \geq G(f, R)$, $M_1(f, R) \geq G_1(f, R)$, $M_2(f, R) \geq G_2(f, R)$, and $A(f) = H(f) = K(f)$.

4. We now prove that A is lower semi-continuous on $\mathcal{C}$ with respect to convergence in measure. Convergence in measure has the associated metric $\delta(f, g) = \inf[k: |f(x, y) - g(x, y)| < k \text{ on a set of measure greater than } 1 - k]$. Then $\{f_n\}$ converges in measure to f if and only if $\lim_{n \rightarrow \infty} \delta(f, f_n) = 0$.

We first prove

LEMMA 1. Let $f \in \mathcal{C}$ and $R = [a, b] \times [c, d]$. For every $\epsilon > 0$, there is a $\sigma > 0$, such that if $0 < \rho < \sigma$, there is an $\eta < 0$ such that if $R_\rho = [a - \rho, b + \rho] \times [c - \rho, d + \rho]$, and if $g \in \mathcal{C}$ is such that $\delta(g, f) < \eta$, then

$$M_1(g, R) \geq G_1(f, R) - \epsilon.$$

PROOF. There is a $\delta > 0$ such that if g and h are functions of one variable such that $|g(y) - f(a, y)| < \delta$ on a subset $E_1 \subset [c, d]$ of measure greater than $(d - c) - \delta$, and $|h(y) - f(b, y)| < \delta$ on a subset $E_2 \subset [c, d]$ of measure greater than $(d - c) - \delta$, then

$$\int_E |g(y) - h(y)| dy > G_1(f, R) - \epsilon,$$

where

$$E = E_1 \cap E_2.$$

Choose $\sigma \geq 1/2$, so that $y \in [c, d]$, $x \in [a - \sigma, a]$ implies $|f(x, y) - f(a, y)| < \delta/2$ and $y \in [c, d]$, $x \in [b, b + \sigma]$ implies $|f(x, y) - f(b, y)| < \delta/2$. Let $0 < \rho < \sigma$. Choose $\eta = \delta\rho$. Let $g \in \mathcal{C}$ be such that $\delta(f, g) < \eta$. Then, let

$$E_1 = \{y \in [c, d] : |f(x, y) - g(x, y)| < \eta \text{ for some } x = x_1(y) \in [a - \rho, a]\},$$

and

$$E_2 = \{y \in [c, d] : |f(x, y) - g(x, y)| < \eta \text{ for some } x = x_2(y) \in [b, b + \rho]\}.$$

It follows that $|E_1| > (d-c) - \delta$ and $|E_2| > (d-c) - \delta$. For, otherwise, we would have $|f(x, y) - g(x, y)| \geq \eta$ on a set of measure $\delta\rho = \eta$, in contradiction with our assumption that $\delta(f, g) < \eta$.

Now, for every $y \in E_1$,

$$\begin{aligned} |g(x_1(y), y) - f(a, y)| & \leq |g(x_1(y), y) - f(x_1(y), y)| + |f(x_1(y), y) - f(a, y)| \\ & < \eta + \frac{\delta}{2} = \rho\delta + \frac{\delta}{2} < \sigma + \frac{\delta}{2} \leq \delta. \end{aligned}$$

Similarly, for every $y \in E_2$,

$$|g(x_2(y), y) - f(b, y)| < \delta,$$

so that

$$\int_E |g(x_2(y), y) - g(x_1(y), y)| dy > G_1(f, R) - \epsilon,$$

where $E = E_1 \cap E_2$.

But, it is evident that

$$M_1(g, R_\rho) \geq \int_E |g(x_2(y), y) - g(x_1(y), y)| dy$$

so that

$$M_1(g, R_\rho) > G_1(f, R) - \epsilon.$$

Having proved the lemma, we prove the lower semi-continuity of $A(f)$ on $\mathcal{C}$ with respect to convergence in measure.

Suppose $f \in \mathcal{C}$, and $A(f) < \infty$. Let $\epsilon > 0$. There is a finite set $R_1, \dots, R_n$ of pair-wise disjoint oriented rectangles, such that

$$\sum_{i=1}^n G(f, R_i) > A(f) - \frac{\epsilon}{2}.$$

For every $i = 1, \dots, n$, let $\sigma_i > 0$ be the constant of Lemma 1 corresponding to R_i and $\epsilon/4n$. Let $\rho < \min |\sigma_1, \dots, \sigma_n|$ be such that the rectangles $R_{1\rho}, \dots, R_{n\rho}$ are pair-wise disjoint. Then, there is an η such that if $\delta(f, g) < \eta$, then for every $i = 1, \dots, n$,

$$M_1(g, R_{i\rho}) > G_1(f, R_i) - \frac{\epsilon}{4n},$$

$$M_2(g, R_{i\rho}) > G_2(f, R_i) - \frac{\epsilon}{4n},$$

and, since $|R_{i\rho}| > |R_i|$, it follows that $M(g, R_{i\rho}) > G(f, R_i) - \epsilon/2n$. We then have $K(g) \geq \sum_{i=1}^n M(g, R_{i\rho}) > \sum_{i=1}^n G(f, R_i) - \epsilon/2 > A(f) - \epsilon$. Hence $\delta(f, g) < \eta$ implies $A(g) > A(f) - \epsilon$.

Suppose $f \in \mathcal{C}$, and $A(f) = +\infty$. Consider any natural number N . There are pair-wise disjoint oriented rectangles $R_1, \dots, R_n$ such that $\sum_{i=1}^n G(f, R_i) > N + 1$. There is a $\rho > 0$ such that the $R_{i\rho}$, $i=1, \dots, n$, are pair-wise disjoint and an $\eta > 0$ such that if $\delta(f, g) < \eta$ then, for every i , $M_1(g, R_{i\rho}) > G_1(f, R_i) - 1/2n$ and $M_2(g, R_{i\rho}) > G_2(f, R_i) - 1/2n$, so that, as above, $K(g) > \sum_{i=1}^n G(f, R_i) - 1 > N$. Hence $\delta(f, g) < \eta$ implies $A(g) > N$. This proves the

LEMMA 2. *The nonparametric Lebesgue area is lower semi-continuous, on the set of continuous functions, with respect to convergence in measure.*

5. It is now an easy matter to prove the theorem. Let Δ be a metric for $\mathcal{O}$ of the type described in the theorem. Let $p \in \mathcal{O}$ and let $p_n \in \mathcal{O}$, $n=1, 2, \dots$, with $\lim_{n \rightarrow \infty} \Delta(p, p_n) = 0$. Then $\{p_n\}$ converges in measure to p . Since $\mathcal{O} \subset \mathcal{C}$, Lemma 2 implies that $\liminf_{n \rightarrow \infty} E(p_n) \geq E(p)$, so that E is lower semi-continuous on $\mathcal{O}$, metrized by Δ .

Since the topology of Δ is coarser than that of uniform convergence, $\mathcal{O}' \supset \mathcal{O}$. Now, let $f \in \mathcal{C}$. Let $p_n \in \mathcal{O}$, $n=1, 2, \dots$, be such that $\lim_{n \rightarrow \infty} \Delta(p_n, f) = 0$. But this implies that $\{p_n\}$ converges in measure to f , so that by Lemma 2, $\liminf_{n \rightarrow \infty} E(p_n) \geq A(f)$. It follows that $\alpha_\Delta(f) \geq A(f)$. However, since uniform convergence implies convergence in the topology corresponding to the Δ metric, it is evident that $\alpha_\Delta(f) \leq A(f)$. This proves the theorem.

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