ON AFFINELY CONNECTED MANIFOLDS WHOSE TORSION CAN BE TRANSFORMED INTO CONSTANT COMPONENTS¹

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Let $\bar{g}_{ik}(\bar{x}_1, \dots, \bar{x}_n) = \bar{g}_{ki}(\bar{x}_1, \dots, \bar{x}_n)$ be the components of the fundamental tensor of a nonsingular Riemannian space V_n of n dimensions ($n \geq 2$, $|\bar{g}_{ik}| \neq 0$) and

$$\left\{ \begin{matrix} i \\ kl \end{matrix} \right\}^*, \quad \bar{R}_{klm}^i$$

the corresponding Christoffel symbols of the second kind and mixed components of curvature.

In the case of a constant metric $g_{\alpha\beta}(x_1, \dots, x_n)$ with respect to the coordinates $x_1, \dots, x_n$ the following relations hold for the corresponding Christoffel symbols

$$\left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}$$

and components $R_{\beta\gamma\delta}^\alpha$ of curvature

$$(1) \quad \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \equiv 0, \quad R_{\beta\gamma\delta}^\alpha \equiv 0, \quad \{x_1, \dots, x_n\}, \quad \alpha, \beta, \gamma, \delta = 1, 2, \dots, n.$$

Conversely the vanishing of Christoffel symbols

$$\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$$

in the identities (1) and the transformation law of Christoffel symbols require for the transformation $\bar{x} \rightarrow x$ the system of conditions

$$\frac{\partial^2 x_\gamma}{\partial \bar{x}_i \partial \bar{x}_k} = \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} \frac{\partial x_\gamma}{\partial \bar{x}_l}, \quad \gamma, i, k, l = 1, 2, \dots, n.$$

Its conditions of integrability

$$\bar{R}_{klm}^i \equiv 0, \quad \{\bar{x}_1, \dots, \bar{x}_n\}$$

are satisfied by assumption, since the second equations of the identities (1) must be fulfilled as tensor relations in every coordinate system. Hence the well-known result [1]: the components of the funda-

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mental tensor of a nonsingular Riemannian V_n can be transformed into constants, if and only if the Riemann-Christoffel tensor of curvature identically vanishes.

In an n -dimensional space A_n with *asymmetric* affine connection

$$\Gamma_{\alpha\beta}^{\lambda}(x_1, \dots, x_n) \neq \Gamma_{\beta\alpha}^{\lambda}(x_1, \dots, x_n), \quad \lambda, \alpha, \beta = 1, 2, \dots, n$$

the differences

$$t_{\alpha\beta}^{\lambda} = \Gamma_{\alpha\beta}^{\lambda} - \Gamma_{\beta\alpha}^{\lambda} = -t_{\beta\alpha}^{\lambda}$$

form the components of the simplest tensor of A_n , the so-called *torsion tensor*. For this tensor $t_{\alpha\beta}^{\lambda}$ we are dealing in the following with the analogous problem and put the question: under which circumstances can the components $\tilde{t}_{ik}^l$ of the torsion of an A_n be altogether transformed into constants $c_{\alpha\beta}^{\lambda}$, not all vanishing?

1. Associated tensors. From the $n^2(n-1)/2$ components of the torsion tensor $t_{\alpha\beta}^{\lambda}$ of an affinely connected A_n we get by contraction with respect to $\lambda = \alpha$ the associated *torsion vector* [2]

$$(2) \quad t_{\beta} = t_{\lambda\beta}^{\lambda}$$

and by inner multiplication ("Überschiebung") the associated symmetric covariant quadratic *torsion tensor* [2, p. 191]

$$(3) \quad t_{\alpha\beta} = t_{\alpha\mu}^{\lambda} t_{\beta\lambda}^{\mu} = t_{\beta\alpha}, \quad \alpha, \beta, \lambda, \mu = 1, 2, \dots, n.$$

For $n=2$

$$(4) \quad \frac{n^2(n-1)}{2} = 2, \quad t_{12}^1 = t_2, \quad t_{21}^2 = t_1$$

hold. In this case the torsion tensor $t_{\alpha\beta}^{\lambda}$ is already given by the components t_{β} of the torsion vector (2); the quadratic torsion tensor (3) is a singular one, $|t_{\alpha\beta}| = 0$, but its rank is 1, if the case of vanishing torsion, $t_{\alpha\beta}^{\lambda} = 0$, should be excluded. Nevertheless the matrix

$$\|t_{23}^{\lambda}, t_{31}^{\lambda}, t_{12}^{\lambda}\|, \quad \lambda = 1, 2, 3$$

has at least the rank 1 in the case $n=3$, even if the relations $t_{\alpha\beta} = 0$, $t_{\beta} = 0$ are allowed [2, pp. 196–197].

In the general case, $|t_{\alpha\beta}| \neq 0$, as is well known, the contravariant tensor $t^{\alpha\beta}$ can be associated with the covariant tensor $t_{\alpha\beta}$.

2. $n=2$. For $n=2$ the binary Pfaffian

$$(5) \quad \omega = t_1 dx_1 + t_2 dx_2$$

is given by (4). Let the coefficients t_1, t_2 be constants, $t_\beta = c_\beta$. Then the bilinear covariant ω' of the form (5) vanishes identically and the class of ω is 1. Conversely ω can be reduced to the canonical form $\bar{\omega} = 1 \cdot d\bar{x}_1 + 0 \cdot d\bar{x}_2$, if $\omega' \equiv 0$. Thus we get:

(I) *The torsion $t_{\alpha\beta}^\lambda$ of an affinely connected surface A_2 can be transformed into constant components not all zero, precisely if its torsion vector t_β is a gradient vector.*

If a vector ξ^α along a curve $x_i(s)$ in A_2 is autoparallel for all possible choices of the torsion, then ξ^α is necessarily the tangent vector of an autoparallel curve [3]. The differential condition that such a vector (in A_n) must satisfy is $t_{\alpha\beta}^\lambda \xi^\alpha x'^\beta = 0$, as one sees by writing the equations of parallelism in the form $(\xi^i)' + \Gamma_{(\alpha\beta)}^i \xi^\alpha x'^\beta + t_{\alpha\beta}^i \xi^\alpha x'^\beta / 2 = 0$. For $n=2$ this reduces to $t_{12}^i (\xi^1 x'^2 - x'^2 \xi^1) = 0$, for all nonzero t_{12}^i , and therefore $(\xi^1, \xi^2) = k(x'^1, x'^2)$.

3. $n > 2$, $|t_{\alpha\beta}| \neq 0$. In the case $n > 2$, $|t_{\alpha\beta}| \neq 0$, a nonsingular Riemannian space V_n of n dimensions is associated to the affinely connected space A_n with the torsion $t_{\alpha\beta}^\lambda$. The fundamental tensor of this V_n is $t_{\alpha\beta}$. We can use the brackets of Christoffel

$$\left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\}, \quad [\lambda, \alpha\beta]$$

of the tensor $t_{\alpha\beta}$, to define covariant derivatives $t_{\alpha\beta(\gamma)}^\lambda$ of the torsion tensor $t_{\alpha\beta}^\lambda$. We get

$$(6) \quad \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\} = t^{\lambda\tau} [\tau, \alpha\beta] = \frac{1}{2} t^{\lambda\tau} \left(\frac{\partial t_{\alpha\tau}}{\partial x_\beta} + \frac{\partial t_{\beta\tau}}{\partial x_\alpha} - \frac{\partial t_{\alpha\beta}}{\partial x_\tau} \right) = \left\{ \begin{matrix} \lambda \\ \beta\alpha \end{matrix} \right\}$$

and

$$(7) \quad t_{\alpha\beta(\gamma)}^\lambda = \frac{\partial t_{\alpha\beta}^\lambda}{\partial x_\gamma} + \left\{ \begin{matrix} \lambda \\ \alpha\zeta \end{matrix} \right\} t_{\alpha\beta}^\zeta - \left\{ \begin{matrix} \zeta \\ \alpha\gamma \end{matrix} \right\} t_{\zeta\beta}^\lambda - t_{\alpha\zeta}^\lambda \left\{ \begin{matrix} \zeta \\ \beta\gamma \end{matrix} \right\}.$$

If the components $t_{\alpha\beta}^\lambda$ can be reduced to the constant values $c_{\alpha\beta}^\lambda$ in the coordinate system $x_1, \dots, x_n$, the same is true with respect to (3) for the components $t_{\alpha\beta}$ of the associated quadratic torsion tensor.

Therefore the Christoffel brackets

$$\left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\}$$

the corresponding Riemann-Christoffel tensor of curvature

$$(8) \quad R_{\alpha\beta\gamma}^{\lambda} = \frac{\partial \left\{ \begin{smallmatrix} \lambda \\ \alpha\gamma \end{smallmatrix} \right\}}{\partial x_{\beta}} - \frac{\partial \left\{ \begin{smallmatrix} \lambda \\ \alpha\beta \end{smallmatrix} \right\}}{\partial x_{\gamma}} + \left\{ \begin{smallmatrix} \sigma \\ \alpha\gamma \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \lambda \\ \sigma\beta \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} \sigma \\ \alpha\beta \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \lambda \\ \sigma\gamma \end{smallmatrix} \right\}$$

and the tensor $t_{\alpha\beta(\gamma)}^{\lambda}$ simultaneously are zero, if the derivatives $\partial t_{\alpha\beta}^{\lambda}/\partial x_{\gamma}$ vanish. Vice versa: if the tensor $R_{\alpha\beta\gamma}^{\lambda}$ vanishes in any special system of coordinates, the same is true in all coordinate systems; then special coordinate systems $x_1, \dots, x_n$ exist, for which, following (6),

$$\left\{ \begin{smallmatrix} \lambda \\ \alpha\beta \end{smallmatrix} \right\} \equiv 0, \quad \{x_1, \dots, x_n\}.$$

In such coordinate systems, following (7), the relations

$$t_{\alpha\beta(\gamma)}^{\lambda} = \frac{\partial t_{\alpha\beta}^{\lambda}}{\partial x_{\gamma}}$$

hold. Thus we get

$$\frac{\partial t_{\alpha\beta}^{\lambda}}{\partial x_{\gamma}} = 0 \quad \text{or} \quad t_{\alpha\beta}^{\lambda} = c_{\alpha\beta}^{\lambda},$$

if $R_{\alpha\beta\gamma}^{\lambda} \equiv 0$ and $t_{\alpha\beta(\gamma)}^{\lambda} \equiv 0$. Hence,

(II) *The torsion $t_{\alpha\beta}^{\lambda}$ of a n -dimensional affinely connected space A_n ($n \geq 3$) with*

$$|t_{\alpha\beta}| = |t_{\alpha\mu}^{\lambda} t_{\beta\lambda}^{\mu}| \neq 0$$

can be transformed into constant components $c_{\alpha\beta}^{\lambda}$ not all zero, precisely if the Riemann space V_n associated with A_n is a euclidean space E_n and the tensor $t_{\alpha\beta(\gamma)}^{\lambda}$, as given by (7), vanishes.

Suppose $n \geq 3$. $|c_{\alpha\beta}| \neq 0$; if arbitrary autoparallel vectors along an autoparallel curve of A_n remain autoparallel under all possible changes of the torsion, then the paths of the connection are integrals of the Pfaffian system [3, p. 559]

$$c_{\alpha 1}^{\lambda} dx_1 + \dots + c_{\alpha n}^{\lambda} dx_n = 0, \quad \alpha, \lambda = 1, 2, \dots, n.$$

4. $n=3$. **Examples.** For $n=3$ we get $n^2(n-1)/2=9$ components $t_{\alpha\beta}^{\lambda}$, $n(n+1)/2=6$ components $t_{\alpha\beta}$ and 3 components t_{β} . After an euclidean metric

$$(9) \quad t_{11} = 1, t_{12} = 0, t_{13} = -1, t_{22} = 1, t_{23} = 0, t_{33} = 1$$

has been chosen, we put

$$\begin{aligned} t_{12}^1 &= y_1, \quad t_{13}^1 = y_2, \quad t_{23}^1 = y_3; \quad t_{12}^2 = y_4, \quad t_{13}^2 = y_5, \quad t_{23}^2 = y_6; \\ t_{12}^3 &= y_7, \quad t_{13}^3 = y_8, \quad t_{23}^3 = y_9. \end{aligned}$$

Then, following (3), the quadratic system of 6 equations for 9 unknowns $y_1, y_2, \dots, y_9$ becomes:

$$\begin{aligned} t_{11} &= y_4^2 + 2y_5y_7 + y_8^2 = 1, \quad t_{12} = -y_1y_4 - y_7(y_2 - y_6) + y_8y_9 = 0, \\ (10) \quad t_{22} &= y_1^2 - 2y_3y_7 + y_9^2 = 1, \quad -t_{13} = y_1y_5 + y_2y_8 + y_4y_6 + y_5y_9 = 1, \\ t_{33} &= y_2^2 + 2y_3y_6 + y_6^2 = 1, \quad t_{23} = y_1y_2 + y_3(y_4 - y_8) - y_6y_9 = 0. \end{aligned}$$

Let $f(x_1, x_2, x_3)$ be an arbitrary (differentiable) function. We put:

$$\begin{aligned} (11) \quad y_1 &= y_2 = y_4 = y_6 = y_8 = y_9 = \frac{1}{+2^{1/2}}, \\ y_3 &= y_5 = 0, \quad y_7 = f(x_1, x_2, x_3). \end{aligned}$$

The values (11) satisfy the system (10) for arbitrary functions f , i.e. for arbitrarily chosen components t_{12}^3 .

Now we denote by $\Gamma_{(\alpha\beta)}^\lambda(x_1, \dots, x_n)$ an arbitrary *symmetric* linear connection. Then the three dimensional affinely connected spaces A_3 , which correspond to the torsion (11) have the asymmetric connection [2, p. 191]:

$$\begin{aligned} \Gamma_{12}^1 &= \Gamma_{(12)}^1 + \frac{1}{2(2)^{1/2}}, & \Gamma_{13}^1 &= \Gamma_{(13)}^1 + \frac{1}{2(2)^{1/2}}, & \Gamma_{23}^1 &= \Gamma_{(23)}^1, \\ \Gamma_{12}^2 &= \Gamma_{(12)}^2 + \frac{1}{2(2)^{1/2}}, & \Gamma_{13}^2 &= \Gamma_{(13)}^2, & \Gamma_{23}^2 &= \Gamma_{(23)}^2 - \frac{1}{2(2)^{1/2}}, \\ \Gamma_{12}^3 &= \Gamma_{(12)}^3 + \frac{1}{2}f(x_1, x_2, x_3), & \Gamma_{13}^3 &= \Gamma_{(13)}^3 - \frac{1}{2(2)^{1/2}}, & \Gamma_{23}^3 &= \Gamma_{(23)}^3 - \frac{1}{2(2)^{1/2}}, \\ & & & & & \left(\Gamma_{\alpha\beta}^\lambda = \Gamma_{(\alpha\beta)}^\lambda + \frac{1}{2} \epsilon_{\alpha\beta}^\lambda \right). \end{aligned}$$

Since the components (9) of the euclidean metric have been chosen as constants, the corresponding Christoffel brackets vanish and the covariant derivatives $t_{\alpha\beta(\gamma)}^\lambda$ of the torsion tensor are reduced, following (7), to the derivatives $\partial t_{\alpha\beta}^\lambda / \partial x_\gamma$. But the derivatives $\partial t_{12}^3 / \partial x_\gamma$ can only be zero, if $f = \text{const}$. Thus we may say:

(III) *The conditions $R_{\alpha\beta\gamma}^\lambda = 0$ and $t_{\alpha\beta(\gamma)}^\lambda = 0$ of Theorem (II) are independent conditions.*

The conditions for $t_{\alpha\beta}^\lambda = c_{\alpha\beta}^\lambda$, developed in the foregoing in the case

$n > 2$, require that nonsingular Riemann spaces can be associated with the torsion tensor ($|t_{\alpha\beta}| \neq 0$). Regarding singular Riemann spaces we are especially interested in "reducible" singular Riemann spaces whose matrix $\|t_{\alpha\beta}\|$ can be reduced to the canonical form

$$\left\| \begin{array}{cccc} t_{11}, \dots, t_{1m}, & 0, \dots, 0 \\ \vdots & \vdots & \vdots & \vdots \\ t_{m1}, \dots, t_{mm}, & 0, \dots, 0 \\ 0, \dots, 0, & 0, \dots, 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0, \dots, 0, & 0, \dots, 0 \end{array} \right\|,$$

if $d = n - m$ is the defect of the matrix $\|t_{\alpha\beta}\|$. The theory of linear connections in such singular Riemann spaces as developed by A. P. Norden [5] ($d=1$) and C. Jankiewicz [6] ($d>1$) also seems to be interesting in regard to the problem of constant components $t_{\alpha\beta}^\lambda = c_{\alpha\beta}^\lambda$ in such singular cases. It may be examined later on [7].

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