THE AREA OF A NONPARAMETRIC SURFACE[1]

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1. Introduction. Consider a continuous real-valued function f on Euclidean n-space, E_n, and the associated map

$$\bar{f}: E_n \to E_{n+1}, \qquad \bar{f}(x) = (x_1, \cdots, x_n, f(x)) \text{ for } x \in E_n.$$

It will be proved that *for each finitely, rectilinearly triangulable subset W of E_n the n dimensional Lebesgue area of $\bar{f}|W$ equals the n dimensional Hausdorff measure of $\bar{f}(W)$, and that if this measure is finite then $\bar{f}(W)$ is Hausdorff n rectifiable.*

For the special case $n=2$ these results were obtained in [**F1**].

The following notation will be used:

$\mathfrak{L}_m = m$ dimensional Lebesgue measure over E_m.

$K(x, r) = E_m \cap \{y: |y-x| < r\}$ for $x \in E_m$, $r > 0$.

$\alpha(m) = \mathfrak{L}_m[K(x, 1)]$ for $x \in E_m$.

$C(x, r) = E_m \cap \{y: |y-x| \leqq r\}$ for $x \in E_m$, $r > 0$.

$L_m = m$ dimensional Lebesgue area.

$\mathfrak{I}_m^k = k$ dimensional integralgeometric measure over E_m.

$\mathcal{H}_m^k = k$ dimensional Hausdorff measure over E_m.

For $X \subset E_m$, $\psi_m^k(X) =$ the infimum of the sums

$$\sum_{S \in F} \alpha(k) 2^{-k} (\operatorname{diam} S)^k$$

corresponding to all countable coverings F of X.

The terminology adopted here is consistent with [**F1**] and [**F3**], where detailed information concerning the basic concepts may be found.

2. **A covering theorem.** Solving a problem posed by Wendell H. Fleming, William Gustin [**G**] recently proved:

There is a number C_n such that

$$\psi_n^{n-1}(X) \leqq C_n \mathcal{H}_n^{n-1}(\text{Bdry } X)$$

for every bounded open subset X of E_n.

We include here a short variant of Gustin's argument, partly because the following lemma is useful also for other purposes.

LEMMA. *If A and B are compact subsets of E_n such that $A \cup B$ is a convex set with diameter δ, then*

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$$\frac{\mathcal{L}_n(A)}{\delta^n}\cdot\frac{\mathcal{L}_n(B)}{\delta^n} \leqq \alpha(n)\,\frac{\mathcal{H}_n^{n-1}(A \cap B)}{\delta^{n-1}}\,.$$

PROOF OF THE LEMMA. Assume $\delta=1$, let a and b be the characteristic functions of A and B, and whenever $0\neq z\in E_n$ let p_z be the orthogonal projection mapping E_n onto the subspace perpendicular to z. By Fubini's theorem

$$\begin{aligned}
\mathcal{L}_n(A)\cdot\mathcal{L}_n(B) &= \int_{E_n}\int_{E_n} a(x)b(y)\,d\mathcal{L}_n y\,d\mathcal{L}_n x \\
&= \int_{E_n}\int_{E_n} a(x)b(x+z)\,d\mathcal{L}_n z\,d\mathcal{L}_n x \\
&= \int_{|z|\leq 1} \mathcal{L}_n(\{x\colon x\in A \text{ and } x+z\in B\})\,d\mathcal{L}_n z \\
&\leqq \int_{|z|\leq 1} \mathcal{H}_n^{n-1}[p_z(A\cap B)]\,d\mathcal{L}_n z \leqq \alpha(n)\mathcal{H}_n^{n-1}(A\cap B),
\end{aligned}$$

because every segment joining $x\in A$ to $x+z\in B$ meets $A\cap B$.

PROOF OF THE COVERING THEOREM. Each $x\in X$ is the center of a spherical ball $C(x, r)$ with

$$\frac{\mathcal{L}_n[C(x,r)\cap X]}{\alpha(n)r^n} = \frac{1}{2}\,;$$

in fact this ratio depends continuously on $r>0$, equals 1 for small r, and approaches 0 as r approaches ∞. From [**M**, Theorem 3.5] one obtains a sequence of such balls $C(x_i, r_i)$ which are disjoint and for which

$$X \subset \bigcup_{i=1}^{\infty} C(x_i, 5r_i).$$

Applying the lemma with

$$A = C(x_i, r_i) \cap \operatorname{Clos} X \quad\text{and}\quad B = C(x_i, r_i) - X$$

one finds that

$$\left[\frac{\alpha(n)}{2^{n+1}}\right]^2 \leqq \alpha(n)\,\frac{\mathcal{H}_n^{n-1}[C(x_i, r_i)\cap \operatorname{Bdry} X]}{(2r_i)^{n-1}}$$

for each i, hence

$$\sum_{i=1}^{\infty} \alpha(n-1)(5r_i)^{n-1} \leqq \frac{5^{n-1}2^{n+3}\alpha(n-1)}{\alpha(n)} \mathcal{H}_n^{n-1}(\text{Bdry } X).$$

3. **Density ratios.** Let $Y = \text{range } \bar{f}$. It will be shown that

$$\psi_{n+1}^n[Y \cap K(p, r)] \leqq C_n 2^{n/2+2} \mathfrak{F}_{n+1}^n[Y \cap K(p, 5r)]$$

whenever $p \in E_n$ and $r > 0$. Obviously (see [**F1**, 10.3] or [**F2**, 4.1])

$$\psi_{n+1}^n[Y \cap K(p, r)] \leqq 2^{n/2+1} \frac{\alpha(n)}{\alpha(n-1)} r\psi_n^{n-1}(\bar{f}^{-1}[K(p, r)]).$$

Choose a finitely, rectilinearly triangulable set Q for which

$$\bar{f}^{-1}[K(p, 4r)] \subset Q \subset \bar{f}^{-1}[K(p, 5r)],$$

assume Q is nonempty, hence $\bar{f}^{-1}[K(p, 5r)] - Q$ is nonempty, and infer from [**T**, 3.8, 3.10] that

$$L_n(\bar{f} \mid Q) = \mathfrak{F}_{n+1}^n[\bar{f}(Q)] < \mathfrak{F}_{n+1}^n[Y \cap K(p, 5r)].$$

Use [**T**, 3.8] and [**F3**, 6.18] to secure a continuously differentiable real-valued function g, with the associated map

$$\bar{g}: E_n \to E_{n+1}, \qquad \bar{g}(x) = (x_1, \cdots, x_n, g(x)) \qquad \text{for } x \in E_n,$$

such that

$$\begin{aligned} &\bar{f}^{-1}[K(p, r)] \subset \bar{g}^{-1}[K(p, 2r)], \\ &\bar{g}^{-1}[K(p, 3r)] \subset \bar{f}^{-1}[K(p, 4r)], \\ &\mathcal{H}_{n+1}^n[\bar{g}(Q)] = L_n(\bar{g} \mid Q) < \mathfrak{F}_{n+1}^n[Y \cap K(p, 5r)]. \end{aligned}$$

For $2r < t < 3r$ the preceding covering theorem implies

$$\begin{aligned} \psi_n^{n-1}(\bar{f}^{-1}[K(p, r)]) &\leqq \psi_n^{n-1}(\{x: |\bar{g}(x) - p| < t\}) \\ &\leqq C_n \mathcal{H}_n^{n-1}(\{x: |\bar{g}(x) - p| = t\}) \\ &\leqq C_n \mathcal{H}_{n+1}^{n-1}[\bar{g}(Q) \cap \{y: |y - p| = t\}]. \end{aligned}$$

From the Eilenberg inequality ([**E**] or [**F2**, 3.2]) one obtains

$$\begin{aligned} r\psi_n^{n-1}(\bar{f}^{-1}[K(p, r)]) &\leqq C_n \int_{2r}^{3r} \mathcal{H}_{n+1}^{n-1}[\bar{g}(Q) \cap \{y: |y - p| = t\}] dt \\ &\leqq C_n \frac{2\alpha(n-1)}{\alpha(n)} \mathcal{H}_{n+1}^n[\bar{g}(Q)], \end{aligned}$$

whence the initial assertion follows.

For $\mathcal{H}^n_{n+1}$ almost all p in Y it is known from [**F1**, 10.1][2] that

$$\limsup_{r\to 0+} \alpha(n) r^{-n} \psi^n_{n+1}[Y \cap K(p, r)] \geqq 2^{-n},$$

and one may now conclude that

$$\limsup_{s\to 0+} \alpha(n) s^{-n} \mathrm{F}^n_{n+1}[Y \cap K(p, s)] \geqq 5^{-n} C_n^{-1} 2^{-3n/2-2}.$$

4. **Area, measure and rectifiability.** The preceding result implies in conjunction with [**F1**, 3.1] that

$$\mathcal{H}^n_{n+1}(S) \leqq 5^n C_n 2^{3n/2+2} \mathfrak{F}^n_{n+1}(S) \qquad \text{for } S \subset Y;$$

in case $\mathfrak{F}^n_{n+1}(S) < \infty$ it follows from the structure theorems [**F1**, 9.6, 9.7] that S is $\mathcal{H}^n_{n+1}$ rectifiable and

$$\mathcal{H}^n_{n+1}(S) = \mathfrak{F}^n_{n+1}(S).$$

Moreover [**T**, 3.8] shows that

$$L_n(\bar{f} \mid W) = \mathfrak{F}^n_{n+1}[\bar{f}(W)]$$

whenever W is a finitely, rectilinearly triangulable subset of E_n.

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[2] The proof of this lemma should be corrected as follows: On line 4 replace "$\delta/5$" by "δ." Replace lines 10 to 17 by "For each $S \in F$ we choose a point $x(S) \in B \cap S$. Since $\psi(X) = \phi(X)$ whenever $X \subset E_n$ and diam $X < \epsilon$, we have $\phi(B) \leqq \sum_{S\in F} \phi(B \cap S) = \sum_{S\in F} \psi(B \cap S) \leqq \sum_{S\in F} \psi(B \cap C[x(S), \text{ diam } S]) \leqq \sum_{S\in F} \lambda \chi^k_n(C[x(S), \text{ diam } S]) = \lambda 2^k \sum_{S\in F} \chi^k_n(S) \leqq (\lambda 2^k)^{1/2} \phi(B)$." A similar correction is required in the proof of [**F1**, 3.6].