A NOTE ON SETS OF CONSTANT WIDTH

Z. A. MELZAK

1. Let E_n be the n -dimensional Euclidean space and let K be a bounded closed convex set in E_n . For any direction u there are exactly two supporting hyperplanes of K , orthogonal to u , and the distance $\mathfrak{D}_u(K)$ between them is called the width of K in the direction u . By a hyperplane we always mean one of dimension $n-1$.

K is called a set of constant width 1, or briefly of width 1, if $\mathfrak{D}_u(K)=1$ for all u . A sphere $S_r(x)$ denotes the closed solid sphere in E_n of radius r about the center x . $S_r(x)$ is the circum-sphere (the in-sphere) of K if $K \subset S_r(x)$ ($S_r(x) \subset K$) and the radius r is smallest (largest) possible. We shall prove the following

THEOREM 1. *Let $K \subset E_n$ be a set of width 1. Then K has a unique circum-sphere $S_R(p)$, a unique in-sphere $S_r(q)$ and*

- (1) $1 - [n/2(n+1)]^{1/2} \leq r, R \leq [n/2(n+1)]^{1/2},$
- (2) $r + R = 1,$
- (3) $p = q.$

Further, in both estimates of (1) we have either the strict inequality together or equality together, and the latter occurs if and only if K contains a unit regular n -simplex.

In the proof free use will be made of the standard properties of convex sets in general and sets of constant width in particular. For these see the standard reference [1].

2. The existence of a unique circum-sphere $S_R(p)$ follows at once from the observation that the intersection of two spheres of equal radii and distinct centers is a subset of a sphere of smaller radius.

The existence of a unique r such that $S_r(q)$ is the in-sphere of K for some q or q 's follows from a simple compactness argument and we have

$$r = \max_{x \in K} \min_{y \in \mathfrak{B}(K)} |xy|$$

where $\mathfrak{B}(K)$ is the boundary of K . Suppose now that there are two in-spheres $S_r(q_1)$ and $S_r(q_2)$, $q_1 \neq q_2$. It follows then that $S_r(q_3)$ is also an in-sphere for any q_3 on the open segment (q_1q_2) ; therefore the set $V = \mathfrak{B}(K) \cap S_r(q_3)$ is not empty, let $t \in V$. Let also H be the hyperplane through t orthogonal to tq_3 . Then H is a supporting hyperplane both of $S_r(q_3)$ and K . Since H cannot intersect the interiors of $S_r(q_1)$

and $S_r(q_2)$ it follows that H supports both these spheres. If we let $x_i = H \cap S_r(q_i)$ then the straight segment x_1x_2 lies in $\mathcal{B}(K)$. However, the boundary of a set of constant width cannot contain straight segments, or in other words, every point of it is extremal. This leads to a contradiction and so $q_1 = q_2 = q$ say. We have proved the existence and uniqueness of both spheres.

It is a consequence of the definition of constant width that if H_1 and H_2 are two distinct parallel supporting hyperplanes of K and $x_1 = H_1 \cap \mathcal{B}(K)$, $x_2 = H_2 \cap \mathcal{B}(K)$, then $|x_1x_2| = 1$ and x_1x_2 is orthogonal to H_1 and H_2 . Let x, y, z be three collinear points such that $y \in K - \mathcal{B}(K)$ is fixed and $x, z \in \mathcal{B}(K)$ are variable. It follows from the previous observation that if $|xy|$ is minimum then $|yz|$ is maximum and conversely, while in either case $|xz| = 1$. As a consequence we have

$$(4) \quad S_{1-R}(p) \subset K \subset S_{1-r}(q)$$

which implies that $1-R \leq r$ and $R \leq 1-r$, proving (2).

By the theorem of Jung [2; 3] we have

$$R \leq [n/2(n+1)]^{1/2}.$$

This together with (2) implies (1). (3) is a simple corollary since by (4) and (2) $S_r(p) \subset K$ and so by the uniqueness of the in-sphere $p = q$. This concludes the proof of the first part of our theorem.

3. We show next that if K contains a unit regular n -simplex T and has the circum-sphere $S_R(p)$, then $R = [n/2(n+1)]^{1/2}$ and so by (2) $r = 1 - [n/2(n+1)]^{1/2}$. Further, p is the center of T .

Let T have vertices $v_1, \dots, v_{n+1}$ and center o . We denote by xy the vector from x to y and by $|xy| = (xy \cdot xy)^{1/2}$ its length. We have

$$(5) \quad \sum_1^{n+1} ov_j = 0$$

and

$$(6) \quad ov_i \cdot ov_j = c, \quad i \neq j, \quad |ov_i| = k.$$

Squaring (5) and making use of (6) we get

$$k^2 \sum_1^{n+1} 1 + 2c \sum_{i < j} 1 = 0$$

or

$$(7) \quad k^2 + nc = 0.$$

Since $|v_i v_j| = 1$ for $i \neq j$, we also have

$$(ov_i - ov_j) \cdot (ov_i - ov_j) = 1$$

and, multiplying this out,

$$(8) \quad 2k^2 - 2c = 1.$$

Now by (7) and (8)

$$(9) \quad c = ov_i \cdot ov_j = -1/2(n+1), \quad k = [n/2(n+1)]^{1/2}.$$

Suppose now that $p \neq o$, then it is easy to show that for some v_i

$$|pv_i| > |ov_i| = k = [n/2(n+1)]^{1/2}.$$

However, this contradicts Jung's theorem. Therefore $p = o$ and $R = k = [n/2(n+1)]^{1/2}$.

4. It remains to be shown that if a set K of width 1 has the circum-sphere $S_R(o)$ and $R = [n/2(n+1)]^{1/2}$ then K contains a unit regular n -simplex. Since K is convex it suffices to show that K contains the $n+1$ vertices of such simplex. We need first a few lemmas. Let $X(K) = \mathfrak{B}(K) \cap \mathfrak{B}[S_R(o)]$; it is easy to show that $X(K)$ contains at least two distinct points.

LEMMA 1. *If $S_R(o)$ is the circum-sphere of K and $R = [n/2(n+1)]^{1/2}$ then for $x_1, x_2 \in X(K)$, $x_1 \neq x_2$, we have*

$$(10) \quad ox_1 \cdot ox_2 \geq -1/2(n+1).$$

By the definition of $X(K)$ we have

$$ox_1 \cdot ox_1 = ox_2 \cdot ox_2 = R^2 = n/2(n+1)$$

and

$$|x_1 x_2|^2 = (ox_1 - ox_2) \cdot (ox_1 - ox_2) \leq 1.$$

Hence by multiplying out

$$2R^2 - 2ox_1 \cdot ox_2 \leq 1$$

which gives at once (10).

LEMMA 2. *Under the assumptions of Lemma 1 the set $X(K)$ contains at least $n+1$ points.*

Let $X(K) = \{x_1, \dots, x_j\}$. It is easily shown that o lies in the closed convex hull $\mathfrak{C}(X)$ of $X = X(K)$. Therefore there exist numbers $\alpha_1, \dots, \alpha_j$ such that

$$0 \leq \alpha_i, \quad \sum_1^j \alpha_i = 1, \quad \sum_1^j \alpha_i ox_i = 0.$$

Squaring the last equation and observing that $ox_i \cdot ox_i = R^2$ we have

$$R^2 \sum_1^j \alpha_i^2 + 2 \sum_{i < k} \alpha_i \alpha_k ox_i \cdot ox_k = 0,$$

and by Lemma 1 and the hypothesis that $R^2 = n/2(n+1)$, we get

$$(11) \quad \sum_1^j \alpha_i^2 \leq (2/n) \sum_{i < k} \alpha_i \alpha_k.$$

Since $\sum_1^j \alpha_i = 1$ by squaring we get

$$(12) \quad \sum_1^j \alpha_i^2 = 1 - 2 \sum_{i < k} \alpha_i \alpha_k.$$

By comparing (11) and (12) we obtain

$$n/2(n+1) \leq \sum_{i < k} \alpha_i \alpha_k$$

and this together with (12) yields

$$(13) \quad \sum_1^j \alpha_i^2 \leq 1/(n+1).$$

However, given j nonnegative numbers $\alpha_1, \dots, \alpha_j$ whose sum is 1, the sum of their squares attains its minimum $1/j$ for $\alpha_1 = \dots = \alpha_j = 1/j$. Hence by (13) $1/j \leq 1/(n+1)$ and so $j \geq n+1$, which proves the lemma.

Since $o \in \mathcal{C}(X)$ and by Lemma 2 $X = X(K)$ contains at least $n+1$ points, there exists a subset U of X such that $U = \{x_1, \dots, x_{n+1}\}$ and $o \in \mathcal{C}(U)$.

LEMMA 3. *The points $x_1, \dots, x_{n+1}$ are vertices of a unit regular n -simplex inscribed into $S_R(o)$.*

Repeating the procedure of Lemma 2 we have

$$(14) \quad \sum_1^{n+1} \alpha_i ox_i = 0$$

where $0 \leq \alpha_i$ and $\sum_1^{n+1} \alpha_i = 1$. Moreover, reasoning as before, we get

$$\sum_1^{n+1} \alpha_i^2 = 1/(n+1)$$

and therefore

$$\alpha_1 = \alpha_2 = \cdots = 1/(n+1)$$

so that by (14)

$$(15) \quad \sum_1^{n+1} ox_i = 0.$$

Squaring this and recalling that $ox_i \cdot ox_i = R^2 = n/2(n+1)$, we have

$$R^2 \sum_1^{n+1} 1 + 2 \sum_{i < k} ox_i \cdot ox_k = 0$$

or

$$\sum_{i < k} ox_i \cdot ox_k = -n/4.$$

Now, there are $n(n+1)/2$ numbers $ox_i \cdot ox_k$, $i < k$. By the above their sum is $-n/4$ and so their arithmetic mean is $-1/2(n+1)$. Hence by (10) we get finally

$$(16) \quad ox_i \cdot ox_k = -1/2(n+1), \quad i \neq k.$$

Now the lemma follows at once from (15) and (16).

An examination of the proofs of Lemmas 1-3 shows that no use has been made of the property of constant width. In fact, these lemmas together with Jung's theorem can be stated as an independent proposition which is actually an extension of Jung's result:

THEOREM 2. *A set of diameter 1 has a circum-sphere of radius $R \leq [n/2(n+1)]^{1/2}$ and the equality holds if and only if K contains the vertices of a unit regular n -simplex.*

This concludes the proof of Theorem 1.

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Added in proof. The author has recently found out that parts of Theorem 1 have been proved by H. G. Eggleston in his Cambridge University *Tract on Convexity*.

REFERENCES

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