A NOTE ON DIVIDED POWERS IN A HOPF ALGEBRA

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The algebraic structure of a Hopf algebra over a field was established by A. Borel [1] (see Théorème 6.1). The theorem asserts that if a Hopf algebra is finitely generated in each dimension and the field is perfect then it is algebraically isomorphic to a tensor product of monogenic exterior and polynomial algebras (the latter possibly truncated if the characteristic is prime). Under more restrictive assumptions a similar structure theorem is proved in [5] (see Theorem 7.13) in the case where the field is replaced by an integral domain. In this note we shall make use of the latter theorem in the case where the Hopf algebra has a system of divided powers and then apply the result to H -spaces.

1. Twisted and divided powers. Throughout, J will denote a principal ideal domain. Let H be a (unitary) J -module. An element $x \in H$ is *divisible* by $a \in J$, $a \neq 0$, if there is an element $y \in H$ such that $ay = x$; if x is divisible only by units in J we say x is a *divided element*. A *greatest divisor* of x is an element $g(x) \in J$ which divides x and has the property: if $a \in J$ divides x then a divides $g(x)$. If $g(x)$ and $g'(x)$ are greatest divisors of x , then

$$g(x) = ag'(x), \quad g'(x) = a'g(x), \quad a, a' \in J'$$

from which it follows that $(aa' - 1)g(x) = 0$. Since J has no zero divisors, a and a' are units. Thus $g(x)$ is unique to within unit factors.

PROPOSITION 1.1. *If H is a free J -module then:*

(a) *Every nonzero x may be represented*

$$(1.1) \quad x = g(x)\bar{x}$$

where $g(x)$ is a greatest divisor, $\bar{x}$ a divided element, and (1.1) is unique to within unit factors.

(b) *If $\bar{y}$ is a divided element and $az = b\bar{y}$, then a divides b .*

PROOF. (a) The uniqueness follows at once from the preceding discussion. Let $x \neq 0$; we may write $x = \sum a_i x_i$, $a_i \neq 0$, where the x_i are elements of a basis for H . Factoring out the greatest common divisor of the a_i we obtain a representation (1.1).

(b) It suffices to show that b is a greatest divisor of $x = b\bar{y}$. Writing

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(1.1) for x , we see that $g(x) = bc$ for some c in J . Then $b(c\bar{x} - \bar{y}) = 0$, from which $c\bar{x} = \bar{y}$ and hence c is a unit. Thus b is a greatest divisor of x .

Now assume H is a J -algebra with a unit 1. We recall that the *height* of a nonzero element u is the least integer $h(u) \geq 2$ such that $u^{h(u)} = 0$. If no such integer exists we define $h(u) = \infty$. If H is graded and anticommutative and u is a homogeneous element not in the center of H , then u must have odd degree and J characteristic not 2. It follows that $h(u) = 2$.

By a *sequence of twisted powers* for $x \in H$ we shall mean a sequence of linearly independent divided elements $\{x^{[m]}, 0 \leq m < h(x)\}$ such that

$$(1.2) \quad g(x^m)x^{[m]} = x^m,$$

where the $g(x^m)$ are greatest divisors of x^m such that

$$(1.3) \quad g(x^0) = 1, \quad g(x^1) = 1.$$

Thus the $g(x^m)$ are nonzero elements and

$$(1.4) \quad x^{[0]} = 1, \quad x^{[1]} = x.$$

PROPOSITION 1.2. *Let H be a free module and graded J -algebra. If $x \in H^i$ ($i > 0$) is a divided element then it possesses a sequence of twisted powers; any two such sequences for x differ at most by unit factors. The submodule spanned by a sequence of twisted powers is a subalgebra with multiplication given by*

$$(1.5) \quad x^{[r]}x^{[s]} = \begin{cases} 0, & r + s \geq h(x), \\ t_{r,s}(x)x^{[m]}, & r + s = m < h(x), \end{cases}$$

where

$$(1.6) \quad t_{r,s}(x) = \frac{g(x^m)}{g(x^r)g(x^s)}.$$

PROOF. For each m , $0 \leq m < h(x)$, choose a representation (1.2) with (1.3) holding (any two such representations differ by unit factors). Since H is graded it is clear that the $x^{[m]}$ are linearly independent. Now let $0 \leq r < h(x)$, $0 \leq s < h(x)$, and $r + s = m$; then

$$(1.7) \quad x^m = x^r x^s = g(x^r)g(x^s)x^{[r]}x^{[s]}.$$

If $m \geq h(x)$ then $x^m = 0$; hence, it follows that $x^{[r]}x^{[s]} = 0$. Suppose $m < h(x)$. Comparing (1.7) with (1.2) we see that $g(x^r)g(x^s)$ must divide $g(x^m)$ and (1.5) follows.

The sequence $t(x) = \{t_{r,s}(x)\}$ is determined to within unit factors

and will be called the *type* of x . The subalgebra is said to be *pseudo-generated* by x and is clearly independent of the choice of twisted powers; we shall denote it by $J[x, h(x), t(x)]$. We make the following observations:

(i) If $t_{r,s}(x)$ is a unit for all $r+s < h(x)$ we say x is of *ordinary type*. We may evidently choose $x^{[m]} = x^m$, and hence

$$(1.8) \quad J[x, h(x), t(x)] = J[x]/(x^h), \quad h = h(x).$$

(ii) If $t_{r,s}(x) = a_{r,s} \cdot (r, s)$ for $r+s < h(x)$, where the $a_{r,s}$ are units in J and (r, s) denotes the binomial coefficient $(r+s)!/r!s!$ we shall say that x is of *binomial type*. We may adjust the choices of the twisted powers so that $a_{r,s} = 1$. From (1.6) it then follows that we may rewrite (1.2) as

$$(1.2)' \quad m!x^{[m]} = x^m \quad 0 \leq m < h(x).$$

In analogy with (1.8) we shall write

$$(1.8)' \quad J[x, h(x), t(x)] = J\langle x \rangle / (x^h), \quad h = h(x).$$

(iii) More generally, if $t_{r,s}(x)$ divides (r, s) for all $r+s < h(x)$ we shall say that x is of *pseudobinomial type*.

REMARKS. If $h(x) = 2$ then $J[x, h(x), t(x)]$ is the monogenic exterior algebra $\Lambda_J(x)$ and, moreover, is both of ordinary and binomial type.

Let J have prime characteristic p . If x is of binomial type then (1.2)' implies that $h(x) \leq p$. If x is of pseudobinomial type and $h(x) = p$ then x is of both ordinary and binomial type (this follows from the fact that (r, s) is a unit if $r+s < p$).

Let H be a J -module. If X is a subset of H , by its *rational closure* $|X|$ we shall mean the set of all $y \in H$ such that $ay \in X$ for some non-zero $a \in J$. Clearly, $X \subset |X|$, and $X \subset X'$ implies $|X| \subset |X'|$. Moreover, if X is a submodule then so is $|X|$. The following general lemma is readily proved:

LEMMA. Let H be a free finitely generated J -module. A submodule $G \subset H$ is a direct summand if and only if $G = |G|$.

Let H be an J -algebra. Let $P(X)$ denote the subalgebra generated by the elements of X ; then $|P(X)|$ is a subalgebra of H . If $H = |P(X)|$ we call X a *system of pseudogenerators* for H ; it is a *minimal* system if

$$x \notin |P(X - \{x\})| \quad \text{for all } x \in X.$$

If the inclusion $i: X \rightarrow H$ induces an algebra monomorphism

$$\bigotimes_{x \in X} J[x]/(x^{h(x)}) \rightarrow H$$

we say that X is a *special* system. If H is graded, the elements of X are required to be homogeneous. (Note that over a field the concepts of generating and pseudogenerating systems coincide.)

A graded module is said to be of *finite type* if each homogeneous submodule is finitely generated.

PROPOSITION 1.3. *If H is a torsion-free graded J -algebra of finite type then $J[x, h(x), t(x)]$ is a direct summand.*

PROOF. Let u be a nonzero element of $|J[x, h(x), t(x)]|$. Then there is an element $a \in J, a \neq 0$, such that

$$(1.9) \quad au = \sum a_m x^{[m]}, \quad a_m \neq 0, a_m \in J,$$

relative to a chosen sequence of twisted powers for x . Since H is graded we may also write

$$(1.10) \quad au = \sum ab_i u_i \quad b_i \neq 0, b_i \in J,$$

where the u_i are nonzero homogeneous divided elements of different degrees. Comparing (1.9) and (1.10) we see that for each i there is a unique m such that

$$ab_i u_i = c_m a_m x^{[m]},$$

where $c_m \in J$ is a unit. From Proposition 1.1(b) it follows that a divides a_m . Dividing through (1.9) by a , we see that $u \in J[x, h(x), t(x)]$. The proposition now follows from the preceding lemma.

COROLLARY 1.4. *Under the assumptions of Proposition 1.3, $J[x, h(x), t(x)] \otimes J[x, h(x), t(x)]$ is a direct summand of $H \otimes H$.*

Let H be a graded anticommutative J -algebra with the property that $h(x) = 2$ if x has odd degree. Then H is said to possess a *system of divided powers* [3, Exposé 7] if to each element x of positive even degree there corresponds a sequence of elements $\gamma_m(x) \in H$ ($m = 0, 1, 2, \dots$) such that:

$$(1.11) \quad \gamma_0(x) = 1, \quad \gamma_1(x) = x, \quad \deg \gamma_m(x) = m \cdot \deg x,$$

$$(1.12) \quad \gamma_r(x) \gamma_s(x) = (r, s) \gamma_m(x), \quad r + s = m,$$

$$(1.13) \quad \gamma_m(x + y) = \sum_{r+s=m} \gamma_r(x) \gamma_s(y),$$

$$(1.14) \quad \gamma_m(xy) = \begin{cases} 0 & \text{if } m \geq 2 \text{ and } x \text{ and } y \text{ have odd degrees,} \\ x^m \gamma_m(y) & \text{if } x \text{ has even and } y \text{ positive even degree.} \end{cases}$$

By induction one proves readily

$$(1.15) \quad m! \gamma_m(x) = x^m.$$

(Compare (1.2)' and note that if the characteristic of J is a prime p then (1.15) also implies that $h(x) \leq p$.) Now assume H is a free module. If x is divided we may choose a sequence of twisted powers for x . Since each $x^{[m]}$ is a divided element, in view of Proposition 1.1(b) it follows from (1.15) and (1.2) that $m!$ divides $g(x^m)$ and, consequently, that (r, s) divides $t_{r,s}(x)$.

2. Hopf algebras with divided powers. A Hopf algebra consists of a pair (H, Δ) , where H is a graded, anticommutative, and associative J -algebra with unit 1 (we assume $H^i = 0$ if $i < 0$, $H^0 = J \cdot 1$), and $\Delta: H \rightarrow H \otimes H$ is an algebraic homomorphism (called the *coproduct*) such that

$$\Delta(x) - \Delta'(x) \in H^+ \otimes H^+, \quad x \in H, \quad H^+ = \sum_{i \geq 0} H^i,$$

where

$$\Delta'(x) = \begin{cases} a(1 \otimes 1) & \text{if } x = a \cdot 1, a \in J, \\ x \otimes 1 + 1 \otimes x & \text{if } x \in H^+. \end{cases}$$

An element x is said to be *primitive* if $\Delta(x) = \Delta'(x)$. The subset of primitive elements is a submodule which we denote by π_0 . Let π denote the subalgebra generated by the primitive elements; then H is called a primitive Hopf algebra if $H = [\pi]$. We shall make use of the following theorems:

THEOREM A. *Let (H, Δ) be a Hopf algebra of finite type over a field K of characteristic p such that if p is prime then $u^p = 0$ for all $u \in H$. If $\{1, x_1, x_2, \dots, x_i, \dots\}$ is a minimal sequence of generators then algebraically*

$$H \cong \bigotimes_i K[x_i]/(x_i^{h_i}), \quad h_i = h(x_i),$$

where

$$(2.1) \quad h_i = \begin{cases} 2 & \text{if } x_i \text{ is not in the center of } H, \\ \infty & \text{if } x_i \text{ is in the center of } H \text{ and } p = 0, \\ p & \text{if } x_i \text{ is in the center of } H \text{ and } p \neq 0. \end{cases}$$

THEOREM B. *Under the preceding assumptions, if each x_i is primitive then the sequence $\{1, x_1, x_2, \dots, x_i, \dots\}$ is a basis for the primitive subspace π_0 .*

THEOREM C. *Let (H, Δ) be a torsion-free Hopf algebra of finite type over a principal ideal domain J of characteristic p . If $\{x_1,$*

$x_2, \dots, x_i, \dots\}$ is a special sequence of pseudogenerators such that each x_i is primitive and of binomial type and the submodule spanned by the x_i is a direct summand then

$$H \cong \bigotimes_i J\langle x_i \rangle / (x^{h_i}),$$

where each height $h_i = h(x_i)$ satisfies (2.1).

REMARKS. Theorem A is a portion of Borel's Theorem cited above. The assumption in [1] that the field K be perfect is superfluous when $p=0$; if $p \neq 0$ it may be replaced by the above condition that $u^p=0$ for all $u \in H$. The proof of Theorem A in the latter case is then essentially the same as for the case $p=0$. Theorem B is proved in [4] (Corollaries 2.5 and 2.6). Theorem C is a restatement of Theorem 7.13 in [5]. Note that as stated in [5] it is required that the submodule spanned by the set

$$\begin{aligned} X &= \{1, x_1, x_2, \dots, x_i, \dots\} & \text{if } p=0, \\ X' &= \{x_i^{[p^k]}; 0 \leq p^k < h_i, i=1, 2, \dots\} & \text{if } p \neq 0, \end{aligned}$$

be a direct summand of H . In the latter case, however, since each x_i is of binomial type, it follows that $h_i \leq p$ (see remark after (1.2)') and hence $X'=X$. Note also that the x_i are divided elements.

PROPOSITION 2.1. *Let (H, Δ) be a torsion-free Hopf algebra of finite type over a principal ideal domain J . If $x \in H^i$ ($i > 0$) is a primitive divided element then it is of pseudobinomial type.*

PROOF. Choose a sequence of twisted powers for x . If $h(x)=2$ the proposition is trivial. Suppose $h(x) > 2$; then x is in the center of H . Since x is primitive we may write, using the binomial theorem and (1.2),

$$g(x^m)\Delta(x^{[m]}) = \sum_{r+s=m} (r, s)g(x^r)g(x^s)x^{[r]} \otimes x^{[s]}, \quad 1 \leq m < h(x).$$

In view of Corollary 1.4, $g(x^m)$ divides each coefficient in the summation. From (1.6) it then follows that $t_{r,s}(x)$ divides (r, s) .

Let (H, Δ) be a torsion-free Hopf algebra over a principal ideal domain J . Let K be the quotient field of J with grading $K^0=K$ and $K^i=0$ if $i > 0$. On the graded K -algebra $K \otimes H$ we define a coproduct Δ_K by

$$(2.2) \quad \Delta_K(a \otimes u) = a \otimes \Delta(u), \quad a \in K, u \in H.$$

One checks readily that $(K \otimes H, \Delta_K)$ is a Hopf algebra over K . We note:

- (i) $K \otimes \pi_0$ is its primitive subspace.
- (ii) $K \otimes \pi$ is its primitive subalgebra.
- (iii) $K \otimes H$ is primitive if and only if H is primitive.

Since H is torsion-free it follows from (2.2) that $a \otimes u$ is primitive if and only if u is primitive and hence (i) is clear. (ii) is obvious in view of (i). To prove (iii) note that $K \otimes |\pi| = K \otimes \pi$. Thus if H is primitive we have

$$K \otimes H = K \otimes |\pi| = K \otimes \pi.$$

Conversely, if $K \otimes H$ is primitive we have

$$K \otimes \pi = K \otimes H \supset K \otimes |\pi| \supset K \otimes \pi,$$

and hence $H = |\pi|$. The main theorem is as follows:

THEOREM 2.2. *Let (H, Δ) be a torsion-free primitive Hopf algebra of finite type over a principal ideal domain J . Then H has a system of divided powers if and only if*

$$H \cong \bigotimes_i J\langle x_i \rangle / \langle x_i^{h_i} \rangle,$$

where the x_i are primitive homogeneous (divided) elements and the $h_i = h(x_i)$ satisfy (2.1).

PROOF. Suppose H is isomorphic to such a tensor product. If only one pseudogenerator $x = x_1$ occurs, define $\gamma_m(x) = x^{[m]}$; if more than one occurs then the existence of a system of divided powers follows from Theorem 2 in [2].

Conversely, suppose H possesses a system of divided powers. We first show that the primitive submodule π_0 is a direct summand. Let $u \in H$ and $a \in J$, $a \neq 0$, be such that $au \in \pi_0$. Then

$$a[\Delta(u) - \Delta'(u)] = \Delta(au) - \Delta'(au) = 0.$$

Since H is torsion-free, it follows that $\Delta(u) - \Delta'(u) = 0$, that is, $u \in \pi_0$. Thus π_0 is rationally closed and hence a direct summand.

Choose a basis $X = \{1, x_1, x_2, \dots, x_i, \dots\}$ for π_0 with each x_i homogeneous. We assert that the divided elements x_i are of binomial type. Choose a sequence of twisted powers for x_i . By the remark following (1.15) we know that if $r + s < h_i$ then (r, s) divides $t_{r,s}(x_i)$. The converse is a consequence of Proposition 2.1, and the assertion is proved. It remains only to show that the sequence X is a special system of pseudogenerators in order to apply Theorem C and complete the proof of Theorem 2.2.

Let K be the quotient field of J and consider the Hopf algebra

$(K \otimes H, \Delta_K)$ defined above. As a consequence of (i), (ii), and (iii) we know that it is primitive and that the sequence

$$\bar{X} = \{\bar{1}, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_i, \dots\} \quad \bar{1} = 1 \otimes 1, \bar{x}_i = 1 \otimes x_i,$$

is a basis for $K \otimes \pi_0$ and a system of generators for $K \otimes \pi$. We shall prove that it is minimal. Assume that the corresponding sequence of degrees of the x_i is nondecreasing. We proceed by induction on i . Let $i > 1$ and assume

$$(2.3) \quad \bar{x}_j \notin P(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{j-1}) \quad \text{for } 1 < j \leq i.$$

Note that (2.3) is trivial if $i=1$. We shall show that (2.3) holds if we replace i by $i+1$. Suppose not; then

$$(2.4) \quad \bar{x}_{i+1} \in P(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_i).$$

Now the subalgebra $P(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_i)$ is clearly a Hopf algebra under (the restriction of) Δ_K with each $\bar{x}_j$ primitive and of binomial type. Moreover, by the inductive assumption, the sequence $\{\bar{1}, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_i\}$ is a minimal system of generators. Applying Theorem B, we have that the primitive subspace has $\{\bar{1}, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_i\}$ as basis. Since $\bar{x}_{i+1}$ is primitive it follows from (2.4) that

$$\bar{x}_{i+1} = a_1 \bar{x}_1 + a_2 \bar{x}_2 + \dots + a_i \bar{x}_i, \quad a_j \in K.$$

But this contradicts the fact that $\bar{X}$ is a basis for $K \otimes \pi_0$. This completes the induction and the proof of minimality.

Consider the commutative diagram

$$\begin{array}{ccc} \bigotimes_i J[x_i]/(x_i^{h_i}) & \xrightarrow{j} & H \\ \downarrow & & \downarrow \\ \bigotimes_i K[\bar{x}_i]/(\bar{x}_i^{h_i}) & \xrightarrow{k} & K \otimes H, \quad h_i = h(x_i) = h(\bar{x}_i), \end{array}$$

where the vertical maps are the algebra monomorphisms induced by the injections $u \rightarrow 1 \otimes u$, and the horizontal maps are induced by inclusions. By Theorem A, k is an algebra isomorphism; hence j is a monomorphism and the sequence X is a special system of pseudo-generators for H . Theorem 2.2 thus follows from Theorem C.

We have the following application to H -spaces:

THEOREM 2.3. *Let T be an H -space with homotopy-associative and homotopy-commutative multiplication whose integral cohomology $H^*(T, Z)$ is of finite type and torsion-free. Then $H^*(T, Z)$ has a system of divided powers if and only if*

$$H^*(T, Z) \cong \bigotimes_i Z\langle x_i \rangle / (x_i^{h_i}),$$

where the x_i are primitive homogeneous (divided) elements and $h_i = h(x_i)$ is 2 or ∞ according to whether x_i has odd or even degree respectively.

PROOF. Let Q be the rational field; then by Theorem 2.10 in [4], $H^*(T, Q)$ is a primitive Hopf algebra. Since we may identify

$$H^*(T, Q) = Q \otimes H^*(T, Z)$$

and the coproducts Δ_Q and Δ in $H^*(T, Q)$ and $H^*(T, Z)$, respectively, are related by

$$\Delta_Q(a \otimes u) = a \otimes \Delta(u), \quad a \in Q, u \in H^*(T, Z),$$

it follows from (iii) that $H^*(T, Z)$ is primitive. Theorem 2.3 thus follows from Theorem 2.2.

For the sake of completeness we state the following theorem which follows from Theorem 2.2 (above) and Theorems 6.4 and 6.5 in [5].

THEOREM 2.4. *Under the assumptions of Theorem 2.2, H has a system of divided powers if and only if for the dual Hopf algebra H^* we have*

$$H^* \cong \bigotimes_i J[u_i] / (u_i^{h_i})$$

where the u_i are primitive homogeneous elements in 1-1 dual correspondence with the x_i and $h_i = h(u_i) = h(x_i)$.

3. An example. R. Bott proved

$$H_*(\Omega G_2, Z) = Z[u, v, w] / \{u^2 - 2v\}$$

where ΩG_2 is the loop space over the exceptional Lie group G_2 (see [2, p. 60]). It is readily seen that the formulas

$$(3.1) \quad \begin{aligned} u^{[2m]} &= v^m, & g(u^{2m}) &= 2^m, \\ u^{[2m+1]} &= uv^m, & g(u^{2m+1}) &= 2^m \end{aligned}$$

define a sequence of twisted powers for u . In particular, note that u^3 is not divisible by 3! so that (1.15) fails and $H_*(\Omega G_2, Z)$ has no system of divided powers. In terms of twisted powers we may write

$$(3.2) \quad H_*(\Omega G_2, Z) \cong Z[u, h(u), t(u)] \otimes Z[w]$$

where the height $h(u) = \infty$ and the type $t(u)$ is defined by (1.6) and (3.1). Also, using Theorem 6.4 in [5] we may write

$$H^*(\Omega G_2, Z) \cong Z[x, h(x), t(x)] \otimes Z\langle y \rangle$$

where $h(x) = \infty$ and $t(x)$ is defined by the greatest divisors $g(x^n) = n!/g(u^n)$. In particular, $g(x^2) = 1$ so that (1.15) fails for x^2 and $H^*(\Omega G_2, Z)$ has no system of divided powers.

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ANOTHER CUTOPOINT THEOREM FOR PLANE CONTINUA

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If the subcontinuum M of a topological 2-sphere S does not separate S and is *locally connected*, then each pair of points of M , which are not separated in M by a point of M , belongs to the closure of a connected domain (of S) lying in M . This is true because each such pair of points belongs to a simple closed curve J lying in M and one of the complementary domains of J is a subset of M . However, without local connectedness such a simple closed curve may fail to exist. In fact, the proposition would then be false because (to take an extreme case) of the existence of indecomposable subcontinua of S which fail to separate S . While no point of an indecomposable continuum *separates* it, every point of it *cuts* it. Recently I showed [1] that this stronger form of separation (or rather the lack of it) is sufficient to restore the validity of the above proposition in the absence of local connectedness if a certain restriction were placed upon

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