BAER *-SEMIGROUPS[1]

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1. **Introduction.** Modern mathematics is replete with instances of semigroups S which are equipped with involutory antiautomorphisms $*: S \rightarrow S$, two noteworthy examples being multiplicative groups on the one hand, and the multiplicative semigroups of Baer *-rings [1, Chapter III, Definition 2] on the other. In this paper we take the second example cited above as our point of departure, setting forth certain postulates which determine what we will call a Baer *-semigroup, and showing that such semigroups provide a more or less natural "coordinatization" of the orthocomplemented weakly modular lattices employed by Loomis [2] in his version of the dimension theory of operator algebras.

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2. **Baer *-semigroups.** By an *involution semigroup* we mean a multiplicatively written semigroup S equipped with a mapping $*: S \rightarrow S$, (called the *involution*), such that for $x, y \in S$, $(xy)^* = y^*x^*$ and $(x^*)^* = x^{**} = x$. An element $e \in S$ with the property that $e = e^2 = e^*$ will be called a *projection*.

If K is a two sided ideal in the involution semigroup S, i.e., if $SK \subset K$ and $KS \subset K$, then we will call K a *focal ideal* in case it is so that for each element $x \in S$, the set $\{y \mid y \in S \text{ and } xy \in K\}$ is a principal right ideal generated by a projection. A *Baer *-semigroup* is a pair (S, K) consisting of an involution semigroup S and a focal ideal K in S. Whenever no confusion can result, we will refer to S itself as being the Baer *-semigroup, rather than using the more cumbersome expression (S, K).

Henceforth, we will regard the symbol S as representing a Baer *-semigroup with focal ideal K. We denote by $P = P(S)$ the set of all projections in S, and we partially order P by decreeing that for $e, f \in P$, $e \leqq f$ means that $ef = e$, (or, what is the same thing, that $fe = e$).

It is clear that if a principal right ideal I in S is generated by a projection e, then this projection e is uniquely determined by I. Consequently, each element $x \in S$ determines a unique projection x'

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such that $\{y \mid y \in S \text{ and } xy \in K\} = x'S$. We call the mapping $': S \to P$ the *focal mapping* induced by the focal ideal K. One easily verifies that the focal mapping has the following properties: (i) For $e, f \in P$, $e \leqq f$ implies that $f' \leqq e'$, (ii) for $e \in P$, $e \leqq e''$, (iii) for $e \in P$, $e' = e'''$, and (iv) for $a, b \in S$, $ab = a$ implies that $a'' \leqq b''$. Moreover, we remark that for each element $a \in S$, $a = aa''$.

Say that a projection $e \in P$ is *K-closed* if $e = e''$, and denote the set of all K-closed projections in S by $P' = P'(S)$. Notice that P' is exactly the range of the focal mapping $': S \to P$. Furthermore, for each projection $f \in P$, f'' is the smallest K-closed projection containing f.

One noteworthy feature of the focal ideal K is that if $a \in S$ and $aa^* \in K$, then $a \in K$. In fact, if $aa^* \in K$, then $a^* = a'a^*$, so $a = aa' \in K$. One consequence of the fact just proved is that $K = K^*$; for if $a \in K$, then certainly $a^*a = a^*(a^*)^* \in K$, hence $a^* \in K$.

A question which arises naturally from time to time in the development of our theory is whether a given projection does or does not commute with various elements of S. This question can frequently be settled by an appeal to the fact that if the projection $e \in P$ commutes with the element $a \in S$, then the projection e' will also commute with a. Indeed, if $ae = ea$, then $eae' = aee' \in K$, so $ae' = e'ae'$. Also, if $ae = ea$, then $a^*e = ea^*$, and the above argument gives $a^*e' = e'a^*e'$, whence, $e'a = e'ae'$, so $ae' = e'a$.

We remark that the necessary and sufficient condition that every element of P' commutes with every element of S is that K is a radical ideal, i.e., if y is any element of S some positive integral power of which belongs to K, then y belongs to K.

In the following theorem it will come to light that S must contain a multiplicative unit. In general, of course, it is possible for a semigroup to admit more than one right (or more than one left) unit, but this cannot occur in an involution semigroup. Actually, if u is a right (or left) unit in an involution semigroup, then u is a two-sided unit and $u = u^*$. An analogous assertion can be made for right (or left) zeros in an involution semigroup.

THEOREM 1. *If (S, K) is a Baer *-semigroup, then S has a unit* 1, *and* 1 *is the largest projection in $P'(S)$. Moreover,* $1'$ *is the smallest projection in P',* $1'$ *is a central projection in S, and $K = 1'S = S1'$. Consequently, the focal ideal in a Baer *-semigroup is a principal ideal generated by a central projection.*

PROOF. Let k be any element of K, so that for any element $e \in P'$, $ke \in K$, and hence, $e \leqq k'$. It follows that k' is a right unit, hence a unit for S, so we write $k'=1$. The remainder of the theorem is clear as soon as we observe that $1'=1\cdot 1'\in K$.

The following lemma is an easy generalization of an analogous result in the standard theory of Baer *-rings, so its proof is omitted:

LEMMA 2. *Let M be a nonempty subset of S. Then, the set $\{y \mid y\in S$ and $My\subset K\}$ is a principal right ideal generated by a projection if and only if $\{m' \mid m\in M\}$ has an infimum in P. Moreover, if $f=\inf_P\{m' \mid m\in M\}$, then $f\in P'$ and $\{y \mid y\in S$ and $My\subset K\}=fS$.*

Let us agree to call the focal ideal K *complete* in case for each nonempty subset M of S, the set $\{y \mid y\in S$ and $My\subset K\}$ is a principal right ideal generated by a projection. If K is a complete focal ideal, we will call the Baer *-semigroup (S, K) a *complete* Baer *-semigroup.

We are now ready to come to grips with the question of the existence of the infimum in P' of two elements $e, f\in P'$. In the special case in which e commutes with f, it is clear that $\inf_{P'}\{e, f\}$ exists and equals ef. The general case is easily settled as follows: Since $f'ee'\in K$, then $e'\leqq (f'e)'$, so e', hence also $e=e''$, commutes with $(f'e)'$. Consequently, $\inf_{P'}\{(f'e)', e\}$ exists and equals $(f'e)'e$. Since $f'(f'e)'e=f'e(f'e)'\in K$, then, $(f'e)'e\leqq f$, and $(f'e)'e$ is a lower bound in P' for $\{e,f\}$. We assert that $(f'e)'e$ is, in fact, the infimum in P' of e and f. Indeed, if $q\in P'$ and if $q\leqq e, f$, then $f'eq=f'q=f'fq\in K$, i.e., $q\leqq (f'e)'$. Consequently, $q\leqq (f'e)'e$, and we have proved that $\inf_{P'}\{e,f\}$ exists and equals $(f'e)'e$.

Henceforth, we will use the notation $e\wedge f=(f'e)'e$ for the infimum in P' of the elements $e, f\in P'$. It is immediate that for $e,f\in P'$, the projection $(e'\wedge f')'=[(fe')'e']'$ is the supremum in P' of e and f, and we will accordingly write this supremum as $e\vee f=(e'\wedge f')'$. It follows from these considerations that P' is a lattice with greatest element 1 and smallest element $1'$, and that the mapping $e\rightarrow e'$ from P' onto P' provides the lattice P' with an orthocomplementation.

In [2], Loomis calls an orthocomplemented lattice L *weakly modular* in case $e,f\in L$ with $e\leqq f$ implies that $f=e\vee(f\wedge e')$. We observe that our lattice P' is automatically weakly modular. In fact, for $e,f\in P'$ with $e\leqq f$, we have $f=[(fe')'e']'=[(fe')'\wedge e']'=(fe')''\vee e$. Since f commutes with e, it also commutes with e', hence, $fe'=f\wedge e'=(fe')''$, proving that $f=(f\wedge e')\vee e$.

We observe in passing that a subset N of P' has an infimum in P if and only if it has an infimum in P', and that the two infima, if they exist, must coincide. An analogous assertion cannot be made for suprema.

The following theorem constitutes a summary of the most important results obtained so far:

THEOREM 3. *Let* (S, K) *be a Baer *-semigroup with induced focal mapping* $x \rightarrow x'$. *Then,* S *has a unit* 1, $1'$ *is a central projection, and the focal ideal* K *is a principal two-sided ideal generated by the projection* $1'$. *Moreover, the set* P' *of* K*-closed projections in* S *forms an orthocomplemented weakly modular lattice with* $e \rightarrow e'$ *as orthocomplementation. The lattice* P' *is complete if and only if* K *is a complete focal ideal, i.e., if and only if* (S,K) *is a complete Baer *-semigroup.*

In the following section we will show that given any orthocomplemented weakly modular lattice L, we can always find a Baer *-semigroup (S,K) whose lattice of K-closed projections is isomorphic to L. Thus, it turns out that the orthocomplemented weakly modular lattices can be characterized as those lattices which arise as lattices of K-closed projections in Baer *-semigroups.

3. **Orthocomplemented weakly modular lattices.** In the present section, the symbol L will always represent an orthocomplemented weakly modular lattice with orthocomplementarion $e \rightarrow e'$. A mapping $\phi: L \rightarrow L$ will be said to be *monotone* in case $e,f \in L$ with $e \leqq f$ implies that $e\phi \leqq f\phi$. We will denote by $M(L)$ the semigroup (under function composition) of all monotone maps on L. Borrowing some nomenclature from Halmos [**3**, p. 231], we will call a mapping $\phi: L \rightarrow L$ a *hemimorphism* of L in case $(e \vee f)\phi = e\phi \vee f\phi$ for $e,f \in L$ and $0\phi = 0$. We remark that a hemimorphism ϕ of L is automatically monotone and that it is also submultiplicative, i.e., $(e \wedge f)\phi \leqq e\phi \wedge f\phi$ for $e,f \in L$.

Given two elements ϕ,ϕ^* of $M(L)$, we will say that ϕ and ϕ^* are *mutually adjoint* in case the inequalities $(e'\phi)'\phi^* \leqq e$ and $(e'\phi^*)'\phi \leqq e$ hold for every element $e \in L$. We claim that if $\phi,\phi^*,\phi^+ \in M(L)$, and if both ϕ and ϕ^*, as well as ϕ and ϕ^+, are mutually adjoint, then $\phi^* = \phi^+$. In fact, let e be any element of L and put $f = e\phi^*$. Then, $f'\phi = (e\phi^*)'\phi \leqq e'$, i.e., $e \leqq (f'\phi)'$, hence $e\phi^+ \leqq (f'\phi)'\phi^+ \leqq f = e\phi^*$. Similarly, $e\phi^* \leqq e\phi^+$, so $e\phi^* = e\phi^+$. It follows that $\phi^* = \phi^+$.

Denote by $S(L)$ the subset of $M(L)$ consisting of all those monotone maps ϕ such that there exists at least one, hence exactly one, monotone map ϕ^* with the property that ϕ and ϕ^* are mutually adjoint. It is clear that if $\phi \in S(L)$, then $\phi^* \in S(L)$ and $\phi^{**} = \phi$.

THEOREM 4. $S(L)$ *is an involution semigroup (under function composition) with involution* $\phi \rightarrow \phi^*$. $S(L)$ *has a zero element and every element* $\phi \in S(L)$ *is a hemimorphism of the lattice* L.

PROOF. Let $\phi,\psi \in S(L)$, and let $e \in L$. Then, $(e\phi\psi)'\psi^*\phi^* \leqq (e\phi)'\phi^* \leqq e'$ and $(e\psi^*\phi^*)'\phi\psi \leqq (e\psi^*)'\psi \leqq e'$, proving that $(\phi\psi)^* = \psi^*\phi^*$. The constant mapping $e \rightarrow 0$, (henceforth denoted by the symbol 0), serves as a zero element for $S(L)$. Finally, let e,f be arbitrary elements in L and put $g=e\vee f$. If $\phi\in S(L)$, then, since ϕ is monotone, $e\phi\leqq g\phi$, $f\phi\leqq g\phi$. But, if $h\in L$ is such that $e\phi,f\phi\leqq h$, then, $h'\leqq(e\phi)'$, $(f\phi)'$ and $h'\phi^*\leqq(e\phi)'\phi^*,(f\phi)'\phi^*$. It follows that $h'\phi^*\leqq e',f'$, i.e., that $e,f\leqq(h'\phi^*)'$. Consequently, $g\leqq(h'\phi^*)'$, so $g\phi\leqq(h'\phi^*)'\phi\leqq h$. This proves that $(e\vee f)\phi=e\phi\vee f\phi$. Finally, let us prove that for $e\in L$, $e\phi=0$ if and only if $e\leqq(1\phi^*)'$. Indeed, if $e\phi=0$, then $1\phi^*=(e\phi)'\phi^*\leqq e'$, so $e\leqq(1\phi^*)'$. Conversely, $e\leqq(1\phi^*)'$ implies $e\phi\leqq(1\phi^*)'\phi\leqq 1'=0$, hence $e\phi=0$. In particular, then, $0\phi=0$, so ϕ is a hemimorphism.

LEMMA 5. *Let $\phi,\psi\in S(L)$. Then, $\phi\psi=0$ if and only if $1\phi\leqq(1\psi^*)'$.*

PROOF. If $1\phi\leqq(1\psi^*)'$, then $e\in L$ implies $e\phi\psi\leqq 1\phi\psi=0$, so $\phi\psi=0$. Conversely, if $\phi\psi=0$, then $(1\phi)\psi=0$, so $1\phi\leqq(1\psi^*)'$.

For each element $g\in L$, we now define a mapping $\phi_g\in M(L)$ in accordance with the prescription $e\phi_g=(e\vee g')\wedge g$ for $e\in L$. We will prove that ϕ_g is a projection in the involution semigroup $S(L)$. To this end, we first notice that for $h,g\in L$, $h\leqq g$ implies that $h=h\phi_g$, because $h\leqq g$ implies $g'\leqq h'$, and the weak modularity of L gives $h'=(h'\wedge g)\vee g'$, i.e., $h=(h\vee g')\wedge g=h\phi_g$. It follows immediately that $\phi_g=\phi_g^2$. Moreover, for $e\in L$, $(e\phi_g)'\phi_g=[(e\vee g')\wedge g]'\phi_g=[(e'\wedge g)\vee g']\phi_g$ $=[(e'\wedge g)\vee g']\wedge g=(e'\wedge g)\phi_g=e'\wedge g\leqq e'$; hence, ϕ_g^* exists and equals ϕ_g.

We have now assembled all the information needed to prove the following theorem, which is the main theorem of the paper:

THEOREM 6. *If L is any orthocomplemented weakly modular lattice, then $(S(L),\ \{0\})$ is a Baer *-semigroup, and the correspondence $g\leftrightarrow\phi_g$ between L and $P'(S(L))$ is an orthocomplementation preserving lattice isomorphism.*

PROOF. Let $\phi\in S(L)$, and put $g=(1\phi)'$. By Lemma 5, $\phi\phi_g=0$. On the other hand, suppose that $\psi\in S(L)$ is such that $\phi\psi=0$. Again by Lemma 5, we have $1\psi^*\leqq g$, hence for $e\in L$, $e\psi^*\leqq g$, so $e\psi^*\phi_g=e\psi^*$, and consequently, $\psi^*\phi_g=\psi^*$, i.e., $\psi=\phi_g^*\psi=\phi_g\psi$. It follows that $\{\psi\mid\phi\psi=0\}$ is a principal right ideal in $S(L)$ generated by the projection ϕ_g, i.e., that $\{0\}$ is a focal ideal in $S(L)$. It is evident that for $e,f\in L$, $e\leqq f$ if and only if $\phi_e\phi_f=\phi_e$, so that the correspondence $g\leftrightarrow\phi_g$ between L and $P'(S(L))$ is a lattice isomorphism. Furthermore, $(\phi_g)'=\phi_{g'}$, so that this lattice isomorphism preserves orthocomplementation.

In [4], von Neumann has shown that if L is an orthocomplemented *modular* lattice with four or more independent perspective elements, then there exists a *-regular ring R (uniquely determined up to an isomorphism), called the *coordinate ring* for R, such that L is isomorphic to the lattice of all projections of R. Note that if S represents the multiplicative semigroup of R, then $(S, \{0\})$ is a Baer *-semigroup, and the lattice $P'(S)$ is the lattice of all projections of R; hence $P'(S)$ is isomorphic to the lattice L.

We are thus led to define a *coordinate Baer *-semigroup* for an orthocomplemented weakly modular lattice L to be a Baer *-semigroup $(S, \{0\})$ with the property that the lattice $P'(S)$ is isomorphic to L and the isomorphism in question preserves orthocomplementation. The content of Theorem 6, then, is that *any orthocomplemented weakly modular lattice L possesses at least one coordinate Baer *-semigroup, namely $(S(L), \{0\})$*. The coordinate Baer *-semigroup $S(L)$ is a weak substitute for the coordinate ring of von Neumann in the case in which L is not modular, but only weakly modular.

Incidentally, no use was made of the weak modularity of L up to and including Lemma 5 in the present section, hence the involution semigroup $S(L)$ is available whenever L is any orthocomplemented lattice. It is not difficult to prove that $\{0\}$ is a focal ideal in $S(L)$ if *and only if* L is a weakly modular lattice, but we will not give the "only if" proof in this paper.

4. The natural homomorphism from S into $S(P'(S))$. If L is an orthocomplemented modular lattice, its coordinate ring is determined up to an isomorphism, but this is manifestly not the case for the coordinate Baer *-semigroups of a weakly modular L. Thus, if S is a Baer *-semigroup and $L = P'(S)$, the coordinate Baer *-semigroup $S(L)$ for L need not be isomorphic to S; however, there does exist—as we will demonstrate in the present section—a natural involution preserving homomorphism ϕ from S into $S(L)$.

For each element $x \in S$, define a mapping ϕ_x: $L \to L$ in accordance with the prescription $e\phi_x = (ex)''$ for $e \in L$. We will see that if g is a projection in $P'(S)$, the mapping ϕ_g as just defined coincides with the hemimorphism $\phi_g \in S(L)$ defined in the previous section, so there will be no notational conflict. It is easy to verify that $1'\phi_x = 1'$ and that $e\phi_x \leqq x''$ for every $x \in S$, $e \in L$. The following lemma shows that ϕ_x is a hemimorphism of L:

LEMMA 7. *Let $\{e_\alpha\} \subset L$ and suppose that $e = \bigvee \{e_\alpha\}$ exists in L. Then, for any element $x \in S$, $\bigvee \{e_\alpha \phi_x\}$ exists in L and is equal to $e\phi_x$.*

PROOF. Let K be the focal ideal for S. For any α, $e_\alpha x(ex)' = e_\alpha ex(ex)' \in K$, so $(ex)' \leqq (e_\alpha x)'$ and $e_\alpha \phi_x \leqq e\phi_x$. On the other hand, if $q \in L$ is such that $e_\alpha \phi_x \leqq q$ for every α, then for every α, $e_\alpha x q' \in K$, $q'x^* e \in K$, hence $e_\alpha \leqq (q'x^*)'$. It follows that $e \leqq (q'x^*)'$, $q'x^*e \in K$, $exq' \in K$,

$q' \leqq (ex)'$, and consequently, $e\phi_x \leqq q$.

Now, let $x \in S$, $e \in L$. Note that since $(ex)'x^*e \in K$, then $e \leqq [(ex)'x^*]'$, i.e., $(e\phi_x)'\phi_{x^*} \leqq e'$. The latter inequality shows that $\phi_x \in S(L)$, in fact, that $(\phi_x)^* = \phi_{x^*}$.

THEOREM 8. *For $x,y \in S$, $\phi_{xy} = \phi_x\phi_y$, hence, the mapping ϕ: $S \rightarrow S(L)$ defined by $\phi(x) = \phi_x$ for $x \in S$ is an involution preserving semigroup homomorphism from S into $S(L)$. Moreover, for $e,g \in L$, $e\phi_g = (eg)'' = (e \vee g') \wedge g$.*

PROOF. Let $x,y \in S$, $e \in L$, and put $ex = a$. Then, $ay(a''y)' = aa''y(a''y)' \in K$, so $(ay)'' \leqq (a''y)''$. Also, $a[a''y(ay)'] = ay(ay)' \in K$, hence $a''y(ay)' = a'a''y(ay)' \in K$, and so $(a''y)'' \leqq (ay)''$, proving that $e\phi_{xy} = (e\phi_x)\phi_y$. Finally, the mappings $e \rightarrow (eg)''$ and $e \rightarrow (e \vee g') \wedge g$ are both now known to be projections in the Baer *-semigroup $S(L)$, and elementary calculation reveals that each majorizes the other.

Notice that if the homomorphism ϕ of the previous theorem is restricted to the subset L of S, one obtains an isomorphism of the lattice L onto the lattice of all $\{0\}$-closed projections in $S(L)$. One consequence of this fact is that if e,f are projections in $P'(S)$, then one can decide whether or not e and f commute in S by checking to see whether or not the hemimorphisms ϕ_e and ϕ_f commute in $S(L)$; hence, if L is any orthocomplemented weakly modular lattice, the commutativity of two elements of L in any coordinate Baer *-semigroup for L implies their commutativity in any other coordinate Baer *-semigroup for L. This suggests a natural way in which von Neumann's notion of the center of a complemented modular lattice [4] can be carried over to the case of an orthocomplemented weakly modular lattice. It turns out that the center of such an L is a Boolean algebra, complete if L is complete, and hence that L itself is a Boolean algebra if and only if it is a subsemigroup of every one of its coordinate Baer *-semigroups.

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