METRIC SPACES IN WHICH BLUMBERG'S THEOREM HOLDS

J. C. BRADFORD AND CASPER GOFFMAN

H. Blumberg, [1], showed that for every real function f on the real line R there is a dense set $D \subset R$ such that f is continuous on D relative to D . The purpose of this paper is to characterize the metric spaces in which Blumberg's theorem holds.

Let S be a metric space and E a subset of S . A point $x \in S$ is said to be of the second category relative to E if every sphere of center x contains a subset of E which is of the second category in S . Otherwise, x is said to be of the first category relative to E . S is said to be of homogeneous second category if every $x \in S$ is of the second category relative to S .

We prove the

THEOREM. *If the metric space S is of homogeneous second category then for every real function f on S there is a dense $D \subset S$ such that f is continuous on D relative to D .*

Conversely, if S is not of homogeneous second category there is a real function f on S which is not continuous on any dense $D \subset S$ relative to D .

PROOF. We prove the converse first. Let S be a metric space which is not of homogeneous second category. There is then a sphere $K \subset S$ which is of the first category in S . Let $K = \bigcup_{n=1}^{\infty} K_n$ where each K_n is nowhere dense in S . Let $f(x) = n$ for every $x \in K_n$, and let $f(x) = 0$ for every $x \in S \sim \bigcup_{n=1}^{\infty} K_n$. Let D be dense in S and let $x \in D \cap K$. Then $x \in K_n$, for some n , so that $f(x) = n$. Let K' be any sphere of center x . Since D is dense and K_n is nowhere dense there is a $y \in (D \sim K_n) \cap K'$. But $|f(x) - f(y)| \geq 1$. Hence, f is not continuous on D relative to D . By an obvious modification, the function f may be taken to be bounded.

For the direct part of the theorem, we need several lemmas.

LEMMA 1 (BANACH, [2]). *If S is a metric space, $E \subset S$, and $T \subset E$ is the set of points in E which are of the first category relative to E then T is of the first category in S .*

Let S be a metric space and $E \subset S$. An $x \in S$ is said to be a heavy point relative to E if there is a sphere of center x every point of which is a point of the second category relative to E .

Received by the editors September 30, 1959.

LEMMA 2. *If S is a metric space and $E \subset S$, then $E = A \cup B$, $A \cap B = \emptyset$, where every $x \in A$ is a heavy point relative to E and B is of the first category in S .*

PROOF. Let B_1 consist of the points of E which are of the first category relative to E . Then B_1 is a set of the first category in S . Let B_2 consist of the points of E which are of the second category relative to E but are not heavy relative to E . Then every sphere in S which has points of B_2 contains a subsphere which has no points of B_2 , so that B_2 is nowhere dense in S . Then $B = B_1 \cup B_2$ is of the first category in S and $A = S \sim B$ is such that every $x \in A$ is contained in a sphere in S every point of which is of the second category relative to E .

Let f be a real function on a metric space S . We shall say that f is almost continuous at x if for every open set G containing $f(x)$ the set $E = \{y: f(y) \in G\}$ has x as a heavy point relative to E .

LEMMA 3. *If S is a metric space and f is a real function on S then f is almost continuous at every point of S except at a set of the first category.*

PROOF. Let $\{G_n\}$ be the set of open intervals with rational end points arranged in one-one correspondence with the natural numbers. For every n , let $S_n \subset S$ be the set of points for which $f(x) \in G_n$ and let $E_n \subset S_n$ be the points of S_n which are not heavy points relative to S_n . The sets E_n are all of the first category, and so $E = \bigcup_{n=1}^{\infty} E_n$ is also of the first category. But if f is not almost continuous at x then $x \in E$.

LEMMA 4. *Let S be a metric space of homogeneous second category, f a real function on S , and $\epsilon > 0$. There is a set $\mathfrak{U} = [U_\alpha: \alpha \in A]$ of pairwise disjoint open spheres in S where, for every $\alpha \in A$, $U_\alpha = \{x: d(x, x_\alpha) < r_\alpha\}$ with $0 < r_\alpha < \epsilon$, and, for every $\alpha \in A$, there is a subset $V_\alpha \subset U_\alpha$, with $x_\alpha \in V_\alpha$, such that every $x \in U_\alpha$ is a heavy point relative to V_α , with the following properties:*

- (a) $\bigcup_{\alpha \in A} U_\alpha$ is dense in S ,
- (b) for every $\alpha \in A$, and $x \in V_\alpha$, $|f(x) - f(x_\alpha)| < \epsilon$.

PROOF. Let $R \subset S$ be the set of points for which f is almost continuous. By Lemma 3, and the assumption that S is of homogeneous second category, R is dense in S . Let x'_α , $\alpha < \xi$, be a well ordering of R . Let $x_1 = x'_1$, and let U_1 be a sphere of center x_1 and radius $r_1 < \epsilon$ such that, for a subset V_1 of U_1 , with $x \in V_1$ and with every point of U_1 of second category relative to V_1 , $|f(x) - f(x_1)| < \epsilon$. Let $x_2 = x'_{\alpha_2}$, where x'_{α_2} is the first element of the well ordered set x'_α , $\alpha < \xi$ which

does not belong to the closure, $\overline{U}_1$, of U_1 . Let U_2 be a sphere of center x_2 and radius $r_2 < \epsilon$, such that $U_2 \cap \overline{U}_1 = \emptyset$, and for a subset $V_2 \subset U_2$, with $x_2 \in V_2$, for which every point in U_2 is a point of second category relative to V_2 , $|f(x) - f(x_2)| < \epsilon$. Suppose, for an ordinal η , we have, for every $\beta < \eta$, x_β , U_β , V_β such that U_β is a sphere of center x_β and radius $r_\beta < \epsilon$, $x_\beta \in V_\beta$, every point of U_β is of second category relative to V_β , $|f(x) - f(x_\beta)| < \epsilon$ for every $x \in V_\beta$, and $\text{Cl}(U_{\gamma < \beta} U_\gamma) \cap U_\beta = \emptyset$. Let $x_\eta = x'_{\alpha_\eta}$ be the first element of the well ordered set x'_α , $\alpha < \xi$, which does not belong to $\text{Cl}(U_{\beta < \eta} U_\beta)$. Let U_η be a sphere of center x_η and radius $r_\eta < \epsilon$, such that $\text{Cl}(U_{\beta < \eta} U_\beta) \cap U_\eta = \emptyset$, and such that there is a $V_\eta \subset U_\eta$, $x_\eta \in V_\eta$, with every point of U_η a point of second category relative to V_η , for which $|f(x) - f(x_\eta)| < \epsilon$ for every $x \in V_\eta$. By transfinite induction, x_α , U_α , V_α , $\alpha < \zeta_1$, are obtained which satisfy the conditions of the lemma.

REMARK 1. For each $\alpha < \zeta_1$, V_α is itself a metric space of homogeneous second category, and so Lemma 4 holds for V_α and any $\epsilon' > 0$.

REMARK 2. For every $x \in S$ there is an $\alpha < \zeta_1$ with $d(x, x_\alpha) < 2\epsilon$.

We now prove the theorem. Let x_α , U_α , V_α , $\alpha < \zeta_1$, be the points, spheres, and sets of Lemma 4 corresponding to $\epsilon = 1$ and the metric space S . For every $\alpha < \zeta_1$, let $x_{\alpha\alpha^1}$, $U_{\alpha\alpha^1}$, $V_{\alpha\alpha^1}$, $\alpha^1 < \zeta_2$, with $x_{\alpha^1} = x_\alpha$, be the points, spheres, and sets of Lemma 4 corresponding to $\epsilon = 1/2$ and the metric space V_α . It is possible to choose $x_{\alpha^1} = x_\alpha$ since f is almost continuous at x_α . For every $\alpha < \zeta_1$, $\alpha^1 < \zeta_2$ let $x_{\alpha\alpha^1\alpha^2}$, $U_{\alpha\alpha^1\alpha^2}$, $V_{\alpha\alpha^1\alpha^2}$, $\alpha^2 < \zeta_3$, with $x_{\alpha\alpha^1} = x_{\alpha\alpha^1}$, be the points, spheres, and sets of Lemma 4 corresponding to $\epsilon = 1/3$ and the metric space $V_{\alpha\alpha^1}$.

We continue in this way ad infinitum, obtaining for every natural number n , $x_{\alpha\alpha^1 \dots \alpha^n}$, $U_{\alpha\alpha^1 \dots \alpha^n}$, $V_{\alpha\alpha^1 \dots \alpha^n}$, $\alpha^n < \zeta_{n+1}$, with $x_{\alpha\alpha^1 \dots \alpha^{n-1}} = x_{\alpha\alpha^1 \dots \alpha^{n-1}}$. These are the points, spheres, and sets of Lemma 4 corresponding to $\epsilon = 1/n$ and the metric space $V_{\alpha\alpha^1 \dots \alpha^{n-1}}$. Let

$$D_0 = [x_\alpha: \alpha < \zeta_1], \quad D_1 = [x_{\alpha\alpha^1}: \alpha < \zeta_1, \alpha^1 < \zeta_2], \dots,$$

$$D_n = [x_{\alpha\alpha^1 \dots \alpha^n}: \alpha < \zeta_1, \alpha^1 < \zeta_2, \dots, \alpha^n < \zeta_{n+1}], \dots,$$

and let $D = \bigcup_{n=0}^{\infty} D_n$.

By Remark 2, D is dense in S . Moreover, for every n , for every $x \in D_n$, the saltus of f at x relative to D is less than $1/n$. For, $x = x_{\alpha\alpha^1 \dots \alpha^n}$ for some $\alpha < \zeta_1$, $\alpha^1 < \zeta_2$, $\dots$, $\alpha^n < \zeta_{n+1}$, $D \cap U_{\alpha\alpha^1 \dots \alpha^n} \subset V_{\alpha\alpha^1 \dots \alpha^n}$ and $x \in V_{\alpha\alpha^1 \dots \alpha^n}$ implies $|f(x) - f(x_{\alpha\alpha^1 \dots \alpha^n})| < 1/n$. Now, let $x \in D$. Then $x \in D_n$ for some n . But, evidently, $D_0 \subset D_1 \subset \dots \subset D_m \subset \dots$, so that $x \in D_m$ for every $m > n$. Hence the saltus of f at x , relative to D , is less than $1/m$ for every m . In other words, f is continuous at x relative to D .

REFERENCES

1. H. Blumberg, *New properties of all real functions*, Trans. Amer. Math. Soc. vol. 24 (1922) pp. 113–128.
2. S. Banach, *Théorème sur les ensembles de première catégorie*, Fund. Math. vol. 16 (1930) pp. 395–398.

ABILENE CHRISTIAN COLLEGE AND
PURDUE UNIVERSITY

CORRECTION TO "GALOIS EXTENSIONS"

CARL FAITH

§4 of [1] is devoted to the question of the existence of a normal basis of a certain Galois extension $\mathbb{R}/\mathbb{F}$. Theorem 2 of this section is incorrectly stated. It can be corrected by replacing the condition (3) *the centralizer $\mathbb{F}'$ of $\mathbb{F}$ in $\mathbb{R}$ is simple* by the condition (3') *the subring $V(\mathfrak{G})$ generated by all regular $x \in \mathbb{R}$ effecting inner automorphisms belonging to $\mathfrak{G}$ is simple*. Simplicity of $V(\mathfrak{G})$ (rather than simplicity of $\mathbb{F}'$), together with the original conditions (1) and (2), implies the equality $V(\mathfrak{G}) = \mathbb{F}'$ which is used in the proof. With this understanding the original proof becomes valid. I am grateful to H. Tominaga for pointing this out (in a letter), and for a reference to his paper [2] in which the results of [1] are extended to primary rings. (Cf. another paper [3] by him in connection with [1] and [2].)

REFERENCES

1. Carl Faith, *Galois extensions in which every element with regular trace is a normal basis element*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 222–229.
2. H. Tominaga, *A note on Galois theory of primary rings*, Math. J. Okayama Univ. vol. 8 (1958) pp. 117–123.
3. ———, *On normal basis theorem and strictly Galois extensions*, Math. J. Okayama Univ. vol. 8 (1958) pp. 133–142.

PENNSYLVANIA STATE UNIVERSITY

Received by the editors September 24, 1959.