$$k^{-n}P_n(k) \leq 2^n \zeta^{-1}(n) + 2^n \epsilon + o(1).$$

Since this is true for any ϵ , one has

$$k^{-n}P_n(k) \leq 2^n \zeta^{-1}(n) + o(1),$$

which gives together with (2) and (7) the relation (6).

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ON THE REPRESENTATION OF SEQUENCES AS FOURIER COEFFICIENTS

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If $F \in L_p(-\infty, \infty)$, where $1 < p \leq 2$, then F has a Fourier transform $G \in L_q(-\infty, \infty)$, where p and q are connected, now and henceforth, by the relation

$$(1) \quad p^{-1} + q^{-1} = 1.$$

However, except in the case $p=2$, the collection of transforms does not cover $L_q(-\infty, \infty)$. In a recent paper [3], we gave a characterization of this collection of Fourier transforms.

A similar situation exists in the theory of Fourier series. If $F \in L_p(0, 2\pi)$, where $1 < p \leq 2$, and $c = \{c_n\}$ is the Fourier sequence of F , that is

$$(2) \quad c_n = \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-int} dt,$$

then [5, §9.1], $c \in l_q$. That is

$$\sum_{n=-\infty}^{\infty} |c_n|^q < \infty.$$

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However, the collection of Fourier sequences of functions in $L_p(0, 2\pi)$ does not cover l_q —see [1; 2]. This collection has been characterized however in terms of the $(C, 1)$ summands of the member sequences—see [5, §4.33].

Here we propose to give another characterization, which will be, in a sense, the discrete analogue of our theory for Fourier transforms. That theory depended on certain theorems concerning the representation of functions as Laplace transforms. Our theory here will depend on analogous theorems concerning the representation of sequences as Hausdorff moment sequences—the discrete analogue of Laplace transforms.

In the theory of Hausdorff moment sequences, the functions

$$\lambda_{k,m}(t) = C_{k,m} t^m (1-t)^{k-m}, \quad k = 0, 1, 2, \dots, 0 \leq m \leq k,$$

play a crucial role—see [4, Chapter 3, §2 et. seq.]. In our theory the numbers

$$(3) \quad a_{r,k,m} = C_{k,m} \int_0^1 t^m (1-t)^{k-m} e^{2\pi i r t} dt, \quad r = 0, \pm 1, \pm 2, \dots, \\ k = 0, 1, 2, \dots, \quad 0 \leq m \leq k,$$

which are essentially the Fourier coefficients of the functions $\lambda_{k,m}$, play a correspondingly important role. Our first efforts must be to derive certain properties of these numbers. We summarize these properties in the lemmas below, and then prove our representation theorem.

LEMMA 1.

$$(4) \quad a_{0,k,m} = (k+1)^{-1}.$$

$$(5) \quad a_{r,k,k} = \bar{a}_{r,k,0}.$$

If $r \neq 0$, then

$$(6) \quad a_{r,k,m} = \frac{k}{2\pi i r} (a_{r,k-1,m} - a_{r,k-1,m-1}), \quad 0 < m < k,$$

$$(7) \quad a_{r,k,k} = \frac{k}{2\pi i r} (k^{-1} - a_{r,k-1,k-1}), \quad k > 0,$$

$$(8) \quad a_{r,k,0} = \frac{-k}{2\pi i r} (k^{-1} - a_{r,k-1,0}), \quad k > 0,$$

$$(9) \quad a_{r,k,k} = \begin{cases} (k+1)^{-1}, & r=0, \\ 0, & k=0, r \neq 0, \\ \sum_{n=0}^{k-1} C_{k-1,n} (-1)^n \frac{n!}{(2\pi i r)^{n+1}}, & kr \neq 0. \end{cases}$$

$$(10) \quad a_{r,k,m} = C_{k,m} (-1)^{k-m} \Delta^{k-m} a_{r,m,m}$$

(where Δ is the advancing difference operator operating on the m variable: that is

$$(11) \quad \Delta^k a_m = \sum_{n=0}^k (-1)^n C_{k,n} a_{m+k-n}.$$

PROOF. (4) and (5) are clear. (6), (7), and (8) follow from (3) by integration by parts. The first two parts of (9) are obvious and the last part follows from the second on repeated use of (7). For (10) we have

$$\begin{aligned} a_{r,k,m} &= C_{k,m} \int_0^1 t^m (1-t)^{k-m} e^{2\pi i r t} dt \\ &= C_{k,m} \sum_{n=0}^{k-m} C_{k-m,n} (-1)^{k-m-n} \int_0^1 t^{k-n} e^{2\pi i r t} dt \\ &= C_{k,m} (-1)^{k-m} \sum_{n=0}^{k-m} (-1)^n C_{k-m,n} a_{r,k-n,k-n} \\ &= C_{k,m} (-1)^{k-m} \Delta^{k-m} a_{r,m,m}. \end{aligned}$$

LEMMA 2.

$$(12) \quad |a_{r,k,m}| \leq (k+1)^{-1}.$$

If $r \neq 0$,

$$(13) \quad |a_{r,k,m}| \leq (\pi |r|)^{-1}.$$

PROOF. (12) is obvious. For (13), if $r \neq 0$, $0 < m < k$, we have from (6) and (12),

$$\begin{aligned} |a_{r,k,m}| &\leq \frac{k}{2\pi |r|} (|a_{r,k-1,m}| + |a_{r,k-1,m-1}|) \\ &\leq \frac{k}{2\pi |r|} (k^{-1} + k^{-1}) = (\pi |r|)^{-1}. \end{aligned}$$

Also if $r \neq 0$, $k > 0$, we have from (5), (7) and (12),

$$\begin{aligned} |a_{r,k,k}| &= |a_{r,k,0}| \leq \frac{k}{2\pi|r|} (k^{-1} + |a_{r,k-1,0}|) \\ &\leq \frac{k}{2\pi|r|} (k^{-1} + k^{-1}) = (\pi|r|)^{-1}. \end{aligned}$$

Finally, if $r \neq 0$, then from (9) $|a_{r,0,0}| = 0 \leq (\pi|r|)^{-1}$.

LEMMA 3. If $c \in l_q$, $1 \leq q < \infty$, and $0 \leq m \leq k$, then $\sum_{r=-\infty}^{\infty} c_r a_{r,k,m}$ is convergent.

PROOF. If $q > 1$, then from Hölder's inequality for series, using (12) and (13),

$$\begin{aligned} \sum_{r=-\infty}^{\infty} |c_r a_{r,k,m}| &= \frac{|c_0|}{k+1} + \sum_{r=-\infty}' |c_r a_{r,k,m}| \\ &\leq \frac{|c_0|}{k+1} + \left\{ \sum_{r=-\infty}' |c_r|^q \right\}^{1/q} \left\{ 2 \sum_1^{\infty} \frac{1}{r^p} \right\}^{1/p} < \infty. \end{aligned}$$

If $q = 1$, then

$$\sum_{r=-\infty}^{\infty} |c_r a_{r,k,m}| \leq \frac{1}{k+1} \sum_{r=-\infty}^{\infty} |c_r| < \infty.$$

Let $c \in l_q$, $q \geq 1$, $0 \leq k \leq m$. We can now define

$$\mu_{k,m}(c) \equiv \sum_{r=-\infty}^{\infty} c_r a_{r,k,m}.$$

THEOREM. A necessary and sufficient condition that a sequence $c \in l_q$, $q \geq 2$, be the Fourier sequence of a function in $L_p(0, 2\pi)$ is that there be a constant M so that

$$(k+1)^{p-1} \sum_{m=0}^k |\mu_{k,m}(c)|^p \leq M, \quad k = 0, 1, 2, \dots$$

PROOF OF NECESSITY. Suppose $F \in L_p(0, 2\pi)$, where $1 < p \leq 2$, and

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-int} dt, \quad n = 0, \pm 1, \pm 2, \dots$$

Then

$$F(t) = \mathfrak{L}_p \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad \text{a.e.,}$$

where $\mathfrak{L}_p$ denotes the limit in mean of order p , and if $0 \leq m \leq k$,

$$\begin{aligned}
\mu_{k,m}(c) &= \sum_{r=-\infty}^{\infty} c_r a_{r,k,m} = \lim_{n \rightarrow \infty} \sum_{-n}^n c_r a_{r,k,m} \\
&= \lim_{n \rightarrow \infty} C_{k,m} \sum_{-n}^n c_r \int_0^1 t^m (1-t)^{k-m} e^{2\pi i r t} dt \\
&= \lim_{n \rightarrow \infty} C_{k,m} \int_0^1 t^m (1-t)^{k-m} \left(\sum_{-n}^n c_r e^{2\pi i r t} \right) dt \\
&= C_{k,m} \int_0^1 t^m (1-t)^{k-m} \left(\mathfrak{L}_p \sum_{n \rightarrow \infty}^n c_r e^{2\pi i r t} \right) dt \\
&= C_{k,m} \int_0^1 t^m (1-t)^{k-m} F(2\pi t) dt.
\end{aligned}$$

Hence, using Hölder's inequality

$$\begin{aligned}
|\mu_{k,m}| &\leq C_{k,m} \int_0^1 t^m (1-t)^{k-m} |F(2\pi t)| dt \\
&\leq C_{k,m} \left\{ \int_0^1 t^m (1-t)^{k-m} |F(2\pi t)|^p dt \right\}^{1/p} \left\{ \int_0^1 t^m (1-t)^{k-m} dt \right\}^{1/q} \\
&= (k+1)^{-1/q} \left\{ C_{k,m} \int_0^1 t^m (1-t)^{k-m} |F(2\pi t)|^p dt \right\}^{1/p}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{m=0}^k |\mu_{k,m}|^p &\leq (k+1)^{-p/q} \sum_{m=0}^k C_{k,m} \int_0^1 t^m (1-t)^{k-m} |F(2\pi t)|^p dt \\
&= (k+1)^{-(p-1)} \int_0^1 (t + (1-t))^k |F(2\pi t)|^p dt \\
&= (k+1)^{-(p-1)} \int_0^1 |F(2\pi t)|^p dt,
\end{aligned}$$

and hence there is such an M , namely

$$M = \int_0^1 |F(2\pi t)|^p dt = \frac{1}{2\pi} \int_0^{2\pi} |F(t)|^p dt.$$

PROOF OF SUFFICIENCY. Suppose $c \in l_q$, $q \geq 2$; let

$$\mu_m = \sum_{r=-\infty}^{\infty} c_r a_{r,m},$$

and let

$$\lambda_{k,m} = C_{k,m} (-1)^{k-m} \Delta^{k-m} \mu_m.$$

Then from (10),

$$\begin{aligned}\lambda_{k,m} &= \sum_{r=-\infty}^{\infty} c_r \{C_{k,m}(-1)^{k-m} \Delta^{k-m} a_{r,m,m}\} \\ &= \sum_{r=-\infty}^{\infty} c_r a_{r,k,m} = \mu_{k,m}(c).\end{aligned}$$

Thus

$$(k+1)^{p-1} \sum_{m=0}^k |\lambda_{k,m}|^p \leq M,$$

and hence by [4, Chapter 3, Theorem 5], there is a function $g \in L_p(0, 1)$ such that

$$\mu_m = \int_0^1 t^m g(t) dt, \quad m = 0, 1, \dots$$

Let $F(t) = g(t/2\pi)$. Then $F \in L_p(0, 2\pi)$, and if $\tilde{c}$ is the Fourier sequence of F , then, operating formally, we obtain

$$\begin{aligned}\tilde{c}_n &= \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-in t} dt = \int_0^1 g(t) e^{-2\pi i n t} dt \\ &= \sum_{m=0}^{\infty} \frac{(-2\pi i n)^m}{m!} \int_0^1 t^m g(t) dt = \sum_{m=0}^{\infty} \frac{(-2\pi i n)^m}{m!} \mu_m \\ &= \sum_{m=0}^{\infty} \frac{(-2\pi i n)^m}{m!} \sum_{r=-\infty}^{\infty} c_r a_{r,m,m} \\ &= \sum_{r=-\infty}^{\infty} c_r \sum_{m=0}^{\infty} \frac{(-2\pi i n)^m}{m!} a_{r,m,m} \\ &= \sum_{r=-\infty}^{\infty} c_r \sum_{m=0}^{\infty} \frac{(-2\pi i n)^m}{m!} \int_0^1 t^m e^{2\pi i r t} dt \\ &= \sum_{r=-\infty}^{\infty} c_r \int_0^1 e^{2\pi i (r-n)t} dt = c_n;\end{aligned}$$

that is

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-in t} dt,$$

and c is the Fourier sequence of a function in $L_p(0, 2\pi)$, provided that we justify the two interchanges of integration and summation, and the interchange of order of summation.

The interchanges of integration and summation are easily justified

by the uniform convergence of the exponential series. For the interchange of order of summations, we must show that

$$\mathcal{K}_n = \sum_{r=-\infty}^{\infty} |c_r| \sum_{m=0}^{\infty} \left| \frac{(-2\pi i n)^m}{m!} a_{r,m,m} \right| < \infty, \quad n = 0, \pm 1, \pm 2, \dots$$

But if $n \neq 0$, then from (12) and (13), and using Hölder's inequality,

$$\begin{aligned} \mathcal{K}_n &= |c_0| \sum_{m=0}^{\infty} \frac{(2\pi |n|)^m}{m!} |a_{0,m,m}| + \sum'_{-\infty} |c_r| \sum_{m=0}^{\infty} \frac{(2\pi |n|)^m}{m!} |a_{r,m,m}| \\ &\leq |c_0| \sum_{m=0}^{\infty} \frac{(2\pi |n|)^m}{(m+1)!} + \frac{1}{\pi} \sum'_{-\infty} \frac{|c_r|}{|r|} \sum_{m=0}^{\infty} \frac{(2\pi |n|)^m}{m!} \\ &= |c_0| \frac{e^{2\pi |n|} - 1}{2\pi |n|} + e^{2\pi |n|} \left\{ \sum_{-\infty}^{\infty} |c_r|^q \right\}^{1/q} \left\{ 2 \sum_1^{\infty} r^{-p} \right\}^{1/p} < \infty, \end{aligned}$$

and

$$\mathcal{K}_0 = \sum_{-\infty}^{\infty} |c_r| |a_{r,0,0}| = |c_0|.$$

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