ADDITION THEOREMS FOR ARBITRARY PLANE AND SPHERICAL WAVES¹

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1. Statement of results. An addition theorem for a solution $u(x, y, z, t)$ of the wave equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0,$$

is a representation for u as a sum of partial waves; i.e., as an infinite series each of whose terms solves the wave equation. The purpose of this note is to establish addition theorems for plane waves,

$$u(x, y, z, t) = f(t - z),$$

and spherical waves,

$$u(x, y, z, t) = \frac{f(t - R)}{R}, \quad R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2,$$

in terms of two classes of partial waves associated with the origin

$$r^2 = x^2 + y^2 + z^2 = 0.$$

The first class consists of functions of the form

$$H_n(x, y, z) \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^n \frac{F(t + r) + G(t - r)}{r},$$

where H_n is a harmonic polynomial of degree n . It is well known that such functions solve the wave equation [2, p. 201 ff.; 6, p. 496]. The second class consists of functions of the form

$$P_n \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{F(t + r) + G(t - r)}{r},$$

where P_n is a polynomial. The identity of these classes follows from a theorem on differentiation, due to E. W. Hobson [3, p. 127], which may be stated as follows.

THEOREM. *Let $F(r) \in C^n$ and let $H_n(x, y, z)$ be a harmonic polynomial of degree n . Then*

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$$H_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)F(r) = H_n(x, y, z)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n F(r).$$

The addition theorems that are discussed here are

$$(1.1) \quad f(t-z) = \sum_{n=0}^{\infty} H_n(x^2 + y^2, z) \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \cdot \frac{I^{n+1}f(t+r) - I^{n+1}f(t-r)}{r}$$

and

$$(1.2) \quad \frac{f(t-R)}{R} = \sum_{n=0}^{\infty} H_n(x^2 + y^2, z) \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n a^n \left(\frac{1}{a}\frac{\partial}{\partial a}\right)^n \cdot \frac{I^{2n+1}f(t-r+a) - I^{2n+1}f(t-r-a)}{ra}, \quad r > a,$$

where

$$H_n(x^2 + y^2, z) = (-1)^n \left(n + \frac{1}{2}\right) r^n P_n\left(\frac{z}{r}\right),$$

and $P_n(x)$ is the Legendre polynomial of degree n ,

$$I^m f(t) = \frac{1}{(m-1)!} \int_0^t (t-x)^{m-1} f(x) dx,$$

and

$$R^2 = x^2 + y^2 + (z-a)^2 = r^2 + a^2 - 2az.$$

For fixed t , r and a , the addition theorems are the Legendre expansions of the two wave functions in terms of Legendre polynomials

$$P_n(x), \quad z = rx.$$

It is known, of course, that the wave functions have Legendre expansions. What is new in this note is the proof that the Legendre coefficients of these expansions have the form indicated above. The theorems that are proved here may be stated as follows.

THEOREM 1. *Let $f(x)$ be a function that is defined on $-\infty < x < \infty$ and is Lebesgue integrable on every finite interval. Then*

$$(1.3) \quad \int_{-1}^1 f(t - rx) P_n(x) dx \\ = (-r)^n \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^n \frac{I^{n+1}f(t+r) - I^{n+1}f(t-r)}{r}$$

for all t , all $r > 0$, and $n = 0, 1, 2, \dots$.

THEOREM 2. Under the same hypotheses on f ,

$$(1.4) \quad \int_{-1}^1 \frac{f(t - (r^2 + a^2 - 2rax)^{1/2})}{(r^2 + a^2 - 2rax)^{1/2}} P_n(x) dx \\ = (-r)^n \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^n a^n \left(\frac{1}{a} \frac{\partial}{\partial a} \right)^n \frac{I^{2n+1}f(t-r+a) - I^{2n+1}f(t-r-a)}{ra}$$

for all t , all $r > a$, and $n = 0, 1, 2, \dots$.

After these results are established, conditions under which the expansions (1.1) and (1.2) are valid can be derived from known results on the convergence of the Legendre series of a function. For example, the Riesz-Fischer theorem implies

THEOREM 3. Let $f(x) \in L^2$ on each finite interval of the x -axis. Then for each t and $a > 0$ the equations (1.1) and (1.2) hold on every sphere $r = r_0 > 0$ and $r = r_0 > a$, respectively, the convergence of the series being L^2 (mean square) convergence on the sphere $r = r_0$.

A standard theorem on pointwise convergence of Legendre series, which appears in Kellogg [4, p. 259], implies

THEOREM 4. Let $f(x) \in C^1$ on $-\infty < x < \infty$. Then equations (1.1) and (1.2) hold for each t, z , and $a > 0$, and for $r > 0$ and $r > a$, respectively. The series converge pointwise on this domain of space-time, and converge uniformly on every compact subset.

Further conditions under which the addition theorems are valid may be derived from other results on the convergence of Legendre series; see, e.g., E. W. Hobson [3, Chapter VII].

2. Proof of Theorem 1. The theorem is proved by induction on n . Identical recursion formulas are derived for the functionals on the two sides of equation (1.3). Using these the inductive step is easily justified. The first result is

LEMMA 1. Let f satisfy the hypotheses of Theorem 1. Then the functional

$$\psi_n[f] \equiv \int_{-1}^1 f(t - rx) P_n(x) dx$$

satisfies

$$(2.1) \quad \psi_{n+1}[f] = \frac{2n+1}{r} \psi_n[If] + \psi_{n-1}[f], \quad n = 1, 2, 3, \dots$$

PROOF. Notice that the indefinite integral

$$\int^x f(t - r\xi) d\xi = -\frac{If(t - rx)}{r} + C.$$

Thus

$$\begin{aligned} \psi_{n+1}[f] &= \int_{-1}^1 P_{n+1}(x) d\left(-\frac{If(t - rx)}{r}\right) \\ &= -\frac{1}{r} P_{n+1}(x) If(t - rx) \Big|_{-1}^1 + \frac{1}{r} \int_{-1}^1 If(t - rx) P'_{n+1}(x) dx \end{aligned}$$

by the formula for integration by parts. Together with

$$P_n(1) = 1, \quad P_n(-1) = (-1)^n$$

and the recursion formula [4, p. 127]

$$P'_{n+1} = (2n+1)P_n + P'_{n-1},$$

this implies

$$\begin{aligned} (2.2) \quad \psi_{n+1}[f] &= \frac{(-1)^{n+1} If(t+r) - If(t-r)}{r} + \frac{2n+1}{r} \psi_n[If] \\ &\quad + \frac{1}{r} \int_{-1}^1 If(t - rx) P'_{n-1}(x) dx. \end{aligned}$$

On integrating by parts in the last integral, one finds that the integrated terms cancel the first terms in (2.2) and hence

$$\psi_{n+1}[f] = \frac{2n+1}{r} \psi_n[If] + \int_{-1}^1 f(t - rx) P_{n-1}(x) dx,$$

which is equivalent to (2.1). The integrations by parts are easily justified for integrable f .

The second result that is needed is

LEMMA 2. *Let f satisfy the hypotheses of Theorem 1. Then the functional*

$$\phi_n[f] \equiv (-r)^n \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^n \frac{I^{n+1}f(t+r) - I^{n+1}f(t-r)}{r}$$

satisfies

$$(2.3) \quad \phi_{n+1}[f] = \frac{2n+1}{r} \phi_n[If] + \phi_{n-1}[f], \quad n = 1, 2, 3, \dots$$

PROOF. Notice that the operator

$$O_n = (-r)^n \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^n \frac{1}{r}$$

satisfies the identity [2, p. 209]

$$(2.4) \quad O_n F(r) = \frac{(-1)^n}{r} \sum_{k=0}^n A_{nk} \left(\frac{-1}{r} \right)^k F^{(n-k)}(r), \quad A_{nk} = \frac{(n+k)!}{k!(n-k)!2^k}.$$

This is easily verified by observing that the expansion has the right form and then evaluating the coefficients A_{nk} by substituting a simple function for F . In particular,

$$O_n e^{-r} = \frac{e^{-r}}{r} \sum_{k=0}^n A_{nk} r^{-k} = \frac{e^{-r}}{r} y_n \left(\frac{1}{r} \right)$$

where $y_n(x)$ is a polynomial of degree n . These polynomials are known as the Bessel polynomials and have been studied by Krall and Frink [5] and by Burchall [1]. The property of the polynomials that is needed here is the recursion formula [5]

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x).$$

Now, from (2.4),

$$O_n I^n f(t-r) = \frac{1}{r} \sum_{k=0}^n A_{nk} \frac{I^k f(t-r)}{r^k} = \frac{1}{r} y_n \left(\frac{I}{r} \right) f(t-r)$$

and similarly

$$\begin{aligned} O_n I^n f(t+r) &= \frac{(-1)^n}{r} \sum_{k=0}^n A_{nk} \frac{(-1)^k I^k f(t+r)}{r^k} \\ &= \frac{(-1)^n}{r} y_n \left(\frac{-I}{r} \right) f(t+r). \end{aligned}$$

On combining these with the recursion formula, one gets

$$\begin{aligned}
 (2.5) \quad O_{n+1}I^{n+1}f(t-r) &= \frac{2n+1}{r} \frac{I}{r} y_n \left(\frac{I}{r} \right) f(t-r) + \frac{1}{r} y_{n-1} \left(\frac{I}{r} \right) f(t-r) \\
 &= \frac{2n+1}{r} O_n I^{n+1}f(t-r) + O_{n-1} I^{n-1}f(t-r)
 \end{aligned}$$

and similarly,

$$(2.6) \quad O_{n+1}I^{n+1}f(t+r) = \frac{2n+1}{r} O_n I^{n+1}f(t+r) + O_{n-1} I^{n-1}f(t+r).$$

The identity (2.3) now follows, since

$$\phi_n[f] = O_n I^{n+1}f(t+r) - O_n I^{n+1}f(t-r).$$

In the notation of Lemmas 1 and 2 the conclusion of Theorem 1 is

$$(2.7) \quad \psi_n[f] = \phi_n[f], \quad n = 0, 1, 2, \dots$$

The proof of Theorem 1 is now completed by observing, first, that (2.7) holds for $n=0$ and 1 by direct verification and, second, that Lemmas 1 and 2 imply that (2.7) is true for $n=m+1$ whenever it is true for $n=m$ and $n=m-1$.

3. Proof of Theorem 2. The parameter t plays no essential part in Theorem 2. Moreover,

$$f_-(x) = f(-x) \Rightarrow I^n f_-(x) = (-1)^n I^n f(-x).$$

Hence, Theorem 2 is entirely equivalent to

THEOREM 2'. *Under the hypotheses of Theorem 1,*

$$\begin{aligned}
 (3.1) \quad &\int_{-1}^1 \frac{f((r^2 + a^2 - 2rax)^{1/2})}{(r^2 + a^2 - 2rax)^{1/2}} P_n(x) dx \\
 &= (-r)^n \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^n a^n \left(\frac{1}{a} \frac{\partial}{\partial a} \right)^n \frac{I^{2n+1}f(r+a) - I^{2n+1}f(r-a)}{ra}
 \end{aligned}$$

for $r > a$ and $n = 0, 1, 2, \dots$

It is convenient to prove the theorem in this form. The proof is again by induction. The first step in the induction is

LEMMA 3. *Let f satisfy the hypotheses of Theorem 1. Then the functional*

$$\psi_n[f] \equiv \int_{-1}^1 \frac{f((r^2 + a^2 - 2rax)^{1/2})}{(r^2 + a^2 - 2rax)^{1/2}} P_n(x) dx$$

satisfies

$$(3.2) \quad \psi_{n+1}[f] = \frac{2n+1}{ra} \psi_n[xIf(x)] + \psi_{n-1}[f], \quad n = 1, 2, 3, \dots$$

PROOF. Notice that

$$\int^x \frac{f((r^2 + a^2 - 2ra\xi)^{1/2})}{(r^2 + a^2 - 2ra\xi)^{1/2}} d\xi = -\frac{If((r^2 + a^2 - 2rax)^{1/2})}{ra} + C.$$

Hence, as in the proof of Lemma 1,

$$\begin{aligned} \psi_{n+1}[f] &= \int_{-1}^1 P_{n+1}(x) d\left(-\frac{If((r^2 + a^2 - 2rax)^{1/2})}{ra}\right) \\ &= \frac{(-1)^{n+1}If(r+a) - If(r-a)}{ra} \\ &\quad + \frac{1}{ra} \int_{-1}^1 If((r^2 + a^2 - 2rax)^{1/2}) P'_{n+1}(x) dx \\ &= \frac{(-1)^{n+1}If(r+a) - If(r-a)}{ra} + \frac{2n+1}{ra} \\ &\quad \cdot \psi_n[xIf(x)] + \frac{1}{ra} \int_{-1}^1 If((r^2 + a^2 - 2rax)^{1/2}) P'_{n-1}(x) dx. \end{aligned}$$

On integrating by parts in the last integral, one obtains (3.2).

Using the notation

$$O'_n = (-a)^n \left(\frac{1}{a} \frac{\partial}{\partial a} \right)^n \frac{1}{a},$$

the conclusion of Theorem 2' may be written

$$(3.3) \quad \psi_n[f] = (-1)^n O_n O'_n (I^{2n+1}f(r+a) - I^{2n+1}f(r-a)), \\ n = 0, 1, 2, \dots$$

To complete the proof of (3.3) notice that

$$\begin{aligned} &\psi_{m+1}[f] \\ &= \frac{2m+1}{ra} (-1)^m O_m O'_m (I^{2m+1}[xIf(x)](r+a) - I^{2m+1}[xIf(x)](r-a)) \\ &\quad + (-1)^{m-1} O_{m-1} O'_{m-1} (I^{2m-1}f(r+a) - I^{2m-1}f(r-a)), \end{aligned}$$

by Lemma 3 and the inductive hypothesis that (3.3) holds for $n=m$ and $n=m-1$. Now, a simple calculation gives

$$I^n[xf(x)](t) = -nI^{n+1}f(t) + tI^n f(t).$$

Thus

$$\begin{aligned} \psi_{m+1}[f] &= \frac{2m+1}{ra} (-1)^m O_m O'_m \{ -(2m+1)I^{2m+3}f(r+a) + (r+a)I^{2m+2}f(r+a) \\ &\quad + (2m+1)I^{2m+3}f(r-a) - (r-a)I^{2m+3}f(r-a) \} \\ &\quad + (-1)^{m-1} O_{m-1} O'_{m-1} (I^{2m-1}f(r+a) - I^{2m-1}f(r-a)). \end{aligned}$$

On using the easily verified relations

$$\begin{aligned} O_m r I^n f(r \pm a) &= -r O_{m-1} I^{n-1} f(r \pm a), \\ O'_m a I^n f(r \pm a) &= \mp a O'_{m-1} I^{n-1} f(r \pm a), \end{aligned}$$

one gets

$$\begin{aligned} \psi_{m+1}[f] &= \frac{2m+1}{ra} (-1)^m O'_m \{ -(2m+1)O_m I^{2m+3}f(r+a) - r O_{m-1} I^{2m+1}f(r+a) \\ &\quad + (2m+1)O_m I^{2m+3}f(r-a) + r O_{m-1} I^{2m+1}f(r-a) \} \\ &\quad + \frac{2m+1}{ra} (-1)^m O_m \{ -a O'_{m-1} I^{2m+1}f(r+a) + a O'_{m-1} I^{2m+1}f(r-a) \} \\ &\quad + (-1)^{m-1} O_{m-1} O'_{m-1} (I^{2m-1}f(r+a) - I^{2m-1}f(r-a)). \end{aligned}$$

Combining this with (2.5) and (2.6) gives

$$\begin{aligned} \psi_{n+1}[f] &= \frac{2m+1}{a} (-1)^m O'_m \{ -O_{m+1} I^{2m+3}f(r+a) + O_{m+1} I^{2m+3}f(r-a) \} \\ &\quad + (-1)^{m-1} O'_{m-1} \left\{ \frac{2m+1}{r} O_m I^{2m+1}f(r+a) + O_{m-1} I^{2m-1}f(r+a) \right. \\ &\quad \left. - \frac{2m+1}{r} O_m I^{2m+1}f(r-a) - O_{m-1} I^{2m-1}f(r-a) \right\}. \end{aligned}$$

Another application of (2.5) and (2.6) gives

$$\begin{aligned} \psi_{m+1}[f] &= \frac{2m+1}{a} (-1)^{m+1} O'_m \{ O_{m+1} I^{2m+3}f(r+a) - O_{m+1} I^{2m+3}f(r-a) \} \\ &\quad + (-1)^{m+1} O'_{m-1} \{ O_{m+1} I^{2m+1}f(r+a) - O_{m+1} I^{2m+1}f(r-a) \} \\ &= (-1)^{m+1} O_{m+1} \left\{ \frac{2m+1}{a} O'_m I^{2m+3}f(r+a) + O'_{m-1} I^{2m+1}f(r+a) \right. \\ &\quad \left. - \frac{2m+1}{a} O'_m I^{2m+3}f(r-a) - O'_{m-1} I^{2m+1}f(r-a) \right\}. \end{aligned}$$

Finally, on using the analogues of (2.5) and (2.6) for O'_m , obtained by replacing r by a and t by r in these equations, one gets (3.3) for $n = m + 1$. This completes the induction, since (3.3) can be verified directly for $n = 0$ and $n = 1$.

REFERENCES

1. J. L. Burchnall, *The Bessel polynomials*, Canad. J. Math. vol. 3 (1951) pp. 62-68.
2. A. Clebsch, *Über die Reflexion an einer Kugelfläche*, J. Reine Angew. Math. vol. 61 (1863) pp. 195-262.
3. E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, Cambridge University Press, 1931.
4. O. D. Kellogg, *Foundations of potential theory*, Berlin, J. Springer, 1929.
5. H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 100-115.
6. H. Lamb, *Hydrodynamics*, 6 ed., Cambridge University Press, 1932.

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