ON THE UNIQUENESS OF COMPACT SOLUTIONS FOR CERTAIN ELLIPTIC DIFFERENTIAL EQUATIONS¹

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Summary. Let O be a fixed point in $(n+1)$ -space R^{n+1} , $r=r(p)$ the distance of p from O , $p \in R^{n+1}$, H_ν the ν th mean curvature of an n -dimensional surface in R^{n+1} . In this paper it is proved that the only compact n -surfaces satisfying $r^\nu H_\nu = 1$ are the spheres with center O . Generalizations of this proposition are given in b and c. §a contains two lemmas: Lemma 1 asserts the ellipticity of certain differential equations; in Lemma 2 a strong form of the "maximum principle" (due to E. Hopf) is stated for solutions of elliptic differential equations.

a. Let F be an n -surface in $(n+1)$ -space, $F^n \subset R^{n+1}$, $n \geq 2$, F of class $C^{2,2}$. F has no singularities; self-intersections are allowed; F is connected. The ν th mean curvature is the ν th elementary symmetric function of the principal curvatures $k_1, k_2, \dots, k_n$ ($k_1 \geq k_2 \geq \dots \geq k_n$) divided by $C_{n,\nu}$:

$$(1) \quad \begin{aligned} H_\nu &= C_{n,\nu}^{-1} \sum k_1 k_2 \cdots k_\nu, & \nu &= 1, 2, \dots, n. \\ H_0 &= 1. \end{aligned}$$

We use cartesian coordinates $x^1, x^2, \dots, x^n, x^{n+1}=z$ in R^{n+1} such that F is given by $z=z(x^1, x^2, \dots, x^n)$ in a neighborhood $U(p)$ of the point p , $p \in F$, p =origin of the (x, z) -system. Then H_ν ($\nu \geq 1$) is a differential expression of second order, i.e. an expression in $x=(x^1, x^2, \dots, x^n)$, z , $z_i = \partial z / \partial x^i = \partial_i z$, $z_{ij} = \partial_i \partial_j z$ ($i, j=1, 2, \dots, n$); H_ν is actually a polynomial in the z_i 's, and its coefficients are rational functions of the z_i 's (x and z do not occur). Let $r=r(q) \geq 0$ be the distance of q ($q \in R^{n+1}$) from the fixed point O . $r^\nu H_\nu$ is the ν th reduced mean curvature (cf. [3]). Let c be a constant.

LEMMA 1. (a) $H_1=c$ and $rH_1=c$ ($r>0$) are everywhere absolutely elliptic (partial) differential equations.

(b) If $k_i > 0$ for all $i=1, 2, \dots, n$, then $H_\nu=c$ and $r^\nu H_\nu=c$ ($r>0$) are absolutely elliptic differential equations.

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² In the sequel, F will frequently be a solution of an absolutely elliptic partial differential equation of second order $\Phi=0$ (cf. [2, p. 107-113], for the definitions involved). If F is of class C^3 and if Φ is analytic, then F is analytic by Bernstein's theorem.

PROOF. Let $x = (x^1, x^2, \dots, x^n)$, $x^{n+1} = z$ be coordinates as above, and let F be in contact with the x -plane at p (e.g. in positive contact: $n(p) = e_{n+1}$). We may assume that $x^1, x^2, \dots, x^n$ are local coordinates in $U(p) \subset F$ with the property

$$g_{ij}(p) = \delta_{ij}.$$

Therefore, we get from (cf. [1, §1])

$$|g^{ij}l_{jm} - k\delta_m^i| = 0$$

(eigenvalues $k = k_1, k_2, \dots, k_n$; $i, j, m = 1, 2, \dots, n$),

together with $l_{jm} = \xi_{jm}n$, the relation

$$(2) \quad |(\delta^{ij} + \Delta^{ij}(x)) \cdot z_{jm} \cdot (1 + \Lambda_{jm}(x)) - k\delta_m^i| = 0,$$

where $\Delta^{ij}(p) = 0$ and $\Lambda_{jm}(p) = 0$. (2) implies (according to (1))

$$(3) \quad C_{n,\nu}H_\nu(x) = \sum z_{11}z_{22} \dots z_{\nu\nu} + P_\nu(x, z, z_i, z_{ij}),$$

P_ν polynomial in the z_{ij} 's, whose coefficients vanish for $x = p$.

Using an appropriate numeration of the coordinates x^i , (2) yields

$$(4) \quad z_{jj}(x) = k_j(p) + Q_j(x), \quad Q_j(p) = 0.$$

Let $(H_\nu)_{(ij)} = \partial H_\nu / \partial z_{ij}$. (3) and (4) imply

$$(5) \quad C_{n,\nu}(H_\nu)_{(ij)}(p) = \delta_{ij} \cdot (\sum^{(i)} k_1 \dots \hat{k}_j \dots k_n)(p) = C_{n,\nu} \delta_{ij} \frac{\partial H_\nu}{\partial k_j}(p),$$

where $\hat{k}_j$ means replacement of k_j by 1, and $\sum^{(i)}$ summation over all summands $k_1 k_2 \dots k_n$ in (1), in which k_j occurs. We get from (5)

$$\left. \begin{array}{l} \sum_{i \leq j} (H_\nu)_{(ij)} Y^i Y^j \text{ positive definite} \\ \sum_{i \leq j} (r^\nu H_\nu)_{(ij)} Y^i Y^j \text{ positive definite} \end{array} \right\} \begin{array}{l} \text{for } \nu = 1, \text{ and for } \nu = 2, 3, \dots, n \\ \text{if } k_i > 0 \text{ (for all } i; r > 0). \end{array}$$

This proves (a) and (b).

LEMMA 2. Let F_1, F_2 be two (regular) surfaces in contact at p both of which are solutions of the absolutely elliptic (partial) differential equation (of second order) $\Phi = 0$ (F_1, F_2 of class C^2 ; $\Phi = \Phi(x, z, z_i, z_{ij})$ of class C^0 in all variables $x^1, x^2, \dots, z_{nn}$ and of class C^1 in $z, z_1, z_2, \dots, z_{nn}$). Then the intersection of F_1 and F_2 consists of a set³ N ($p \in N$) such

³ In the analytic case: N is an $(n-1)$ -surface which is in the first approximation a cone with vertex O ; N may have several locally irreducible components, for $n=2$ there must be at least two of them.

that the contact at p between F_1 and F_2 is not semi-proper,⁴ or else F_1 and F_2 coincide in a neighborhood of p .

Proof following [2], with the help of [4] and [5]: the difference of two solutions of $\Phi=0$ solves a homogeneous linear elliptic differential equation (cf. [2, p. 109]); application of [4, Satz 3, p. 149] and of [5] leads immediately to Lemma 2.

b. Let S_0 be the n -sphere in R^{n+1} with center O (and with any (positive) radius).

PROPOSITION 1. Let⁵ F be compact, and let

$$(6) \quad r^\nu H_\nu = 1$$

on F (for a fixed ν ; $1 \leq \nu \leq n$). Then $F = S_0$.

PROOF. Since F is compact, there exists a sphere S_0 which touches F "from outside": take a sphere with center O and a large radius such that the intersection with F is empty, and shrink this sphere homothetically until it touches F for the first time. In the neighborhood of a point of contact we take the inner normals as positive (the inner normal is the positive one on S_0 globally; for odd ν we have to make this choice because (6) holds for S_0 and F). Then the contact is positive, and the principal curvatures of F (and of S_0) are all positive at the points of contact. Lemma 1 for $r^\nu H_\nu = 1$ and Lemma 2 yield coincidence of F with S_0 in a neighborhood of a point of contact, i.e. the set of points of contact is open. By continuity it is also closed, therefore $F = S_0$ (since F is compact and connected).

REMARK. In the proofs of Proposition 1 and later propositions we use only Satz 3 of [4] (maximum principle for linear elliptic differential equations in an open domain), and not the full strength of Lemma 2 (i.e. of [5]) with the notion of semi-proper contact. However, Lemma 2 is useful in the case of surfaces with boundary. We get theorems (with proofs analogous to the one above) similar to Satz 1' and Satz 10' in [3] for a central projection $T: F \rightarrow \bar{F}$ between two surfaces $F, \bar{F}$ with regular boundary and with O as center of the projection T , $O \notin F$ (and therefore $O \notin \bar{F}$ too). We see: all the propositions in this paper possess their counterpart theorems (similar to Satz 1' in [3]) about central projections between surfaces with regular boundary. Parallel projections can be treated the same way (cf. [1]).

⁴ If F_1 and F_2 have a contact at p such that there is on F_1 a regular $(n-1)$ -surface A through p with the property that in a neighborhood of p on at least one side of A , F_1 and F_2 do not intersect, then F_1 and F_2 have a semi-proper contact at p (cf. [2, p. 102]; "intersection" in the strong sense of 3 in [2, p. 102]).

⁵ F of class C^2 , as always.

We assume in the following propositions 1' and 1'' that F is compact and without self-intersections ($R^{n+1} - F = I \cup E$ with $I \cap E$ empty, I is the interior, E the exterior of F , E containing the point at ∞), that $O \in I$, and that the positive normal n_+ is always the inner normal. Furthermore, let the occurring differential equations be taken with respect to this choice of n_+ .

PROPOSITION 1'. *Let*

$$(6') \quad \sum_{r=1}^n a_r r^r H_r = \sum_{r=1}^n a_r > 0$$

on F with nonnegative functions a_r (defined and of class C^1 on $R^{n+1} - O$).⁶ Then $F = S_O$.

The proof is similar to that of Proposition 1, since (6') is absolutely elliptic for $k_i > 0$ ($i = 1, 2, \dots, n$), $a_r \geq 0$, $\sum a_r > 0$.

An analogous argument shows:

PROPOSITION 1''. *Let $\Phi = 0$ be an absolutely elliptic differential equation for $k_i > 0$ ($i = 1, 2, \dots, n$)⁷ such that every sphere S_O is a solution of $\Phi = 0$. Then every solution F (with the properties mentioned above) is a sphere S_O .*

REMARK. Proposition 1 was proved in [3] for $\nu = 1$, $F \in \mathfrak{R}(S, O)$ (i.e. F "star-shaped" in the sense of [3, §3]); furthermore for $\nu = 2, 3, \dots, n$, $F \in \mathfrak{L}(S, O)$. The proofs were based on the following relationships ((3.6) and (6.6) in [3]; $S = n$ -sphere in R^{n+1} with center O and radius 1; κ_n = surface area of S):

$$(7) \quad \int_S r H_1 d\Omega \leq \kappa_n \quad \text{for } F \in \mathfrak{R}(S, O), \quad "=" \Leftrightarrow F = S_O,$$

$$(7_r) \quad \int_S r^r H_r d\Omega \leq \kappa_n \quad \text{for } F \in \mathfrak{L}(S, O), \quad "=" \Leftrightarrow F = S_O,$$

$$\nu = 2, 3, \dots, n.^8$$

c. Let $f(q)$ be a function of class C^1 on S ($q \in S$, S as above), and let $[pq]$ ($p, q \in R^{n+1}$, $p \neq q$) be the straight line in R^{n+1} through p and q . Let the compact surface G have no self-intersections, let n_+ be the inner normal, and let $O \in I$ (I = interior of G).

⁶ a_r may even depend on the normal of F . Then a_r is supposed to be a function of class C^1 on the tangent bundle of $R^{n+1} - O$ (i.e. on the manifold of the elements (v, p) , v unit vector in R^{n+1} , $p \in R^{n+1} - O$).

⁷ Continuity and differentiability of Φ as in Lemma 2.

⁸ I do not know whether (7_r) is true for $F \in \mathfrak{R}(S, O)$.

PROPOSITION 2. *Assume*

$$(8) \quad r(p)H_1(p) = f(q) \quad \text{for } p \in F, q \in S, p \in [Oq],$$

has a solution⁵ F which is star-shaped⁹ with respect to O (and therefore all F_t are solutions, where F_t and F are homothetic with the factor $t > 0$ and with respect to O). Then every solution G (of class C^2 and with the properties mentioned above) is such an F_t .

The proof follows the same lines as that of Proposition 1: to every G there exists an F_t which is in positive contact with G from outside, and therefore $G = F_t$ (for some t) as above. Proposition 2 generalizes Satz 1 in [3] (proved by help of an integral formula) for a star-shaped F .

We may weaken the hypotheses in Proposition 2 because of the following fact: if F is compact, without self-intersections, with $O \in I$, and if F is a solution of (8), then F is star-shaped. To see this, we consider $F' = F_t$ for a sufficiently large $t > 0$ such that $F \cap F'$ is empty; then we contract F' again to a surface F'' with the property that F'' touches F for the first time (F'' is homothetic to F , and F'' touches F from outside). The contact between F and F'' is positive, since $O \in I$ (taking for n_+ the inner normal). (8) implies $F = F''$ by help of the Lemmas 1 and 2 (and because F is compact and connected), and F has to be star-shaped with respect to O , i.e. every half-line starting at O intersects F in exactly one point, otherwise F would have at least two (homothetic) components.

We get easily another proposition which is similar to Proposition 1:

PROPOSITION 2 _{ν} . *Let an ovaloid F with $O \in I$ be a solution of*

$$(8_\nu) \quad r^\nu(p)H_\nu(p) = f(q) > 0, \quad p \in F, q \in S, p \in [Oq],$$

(and therefore all F_t are solutions, $t > 0$). ν is fixed, $1 \leq \nu \leq n$. Then every compact solution is such an F_t .

A priori, the solution of (8 _{ν}) is here allowed to have self-intersections. The proof is again almost the same as above, the surface F_t replacing the spheres S_0 .

For $\nu = n$ every compact solution is automatically an ovaloid (Hadamard) with $O \in I$ ($r > 0$ because $f > 0$). Therefore, Proposition 2 _{n} remains valid if we replace the condition for F to be an ovaloid

⁹ Here and in the sequel, we mean by " F is star-shaped with respect to O " the property that every half-line starting at O intersects F in exactly one point. "Star-shaped" in the sense " $F \in \mathfrak{K}(S, O)$," as used in [3], is a slightly stronger condition. The differential equation (8) is taken with n_+ = inner normal.

by the compactness of F . Proposition 2_v contains Satz 10 in [3], under the additional assumption $O \in I$. Thus Satz 10 (without the hypothesis (c) on the connecting family $\mathfrak{F}$ and leaving out the convexity condition in (b), cf. [3, p. 192]) can be proved for $O \notin F$ using the method explained here: since $\bar{F}$ (in Satz 10) is an ovaloid, the shadow frontier is a regular surface (if not empty); then we apply a proposition, analogous to Proposition 2_v, for a central projection (with center O) between two surfaces F and $\bar{F}$ with regular boundary ($O \notin F$; cf. remark after the proof of Proposition 1).

Let G (of class C^2) in the Propositions 2' and 2'' be again compact, without self-intersections, let $O \in I$, and let n_+ be the inner normal (the differential equations always being taken with n_+ = inner normal).

PROPOSITION 2'. *Let an ovaloid F with $O \in I$ be a solution of*

$$(8') \quad \sum_{v=1}^n a_v r^v(p) H_v(p) = f(q) > 0, \quad p \in F, \quad q \in S, \quad p \in [O q],$$

with $a_v = a_v(q) \geq 0^{10}$ (and therefore all F_t are solutions, $t > 0$). Then every solution G is such an F_t .

PROPOSITION 2''. *Let $\Phi = 0$ be an absolutely elliptic differential equation⁷ with a one-parameter family of compact solutions F_t , $t > 0$, F_t homothetic to $F = F_1$ with the factor t and with respect to O , F without self-intersections, $O \in I$ (hence through any point $p \in R^{n+1}$, $p \neq O$, there passes at least one F_t). Then the surfaces F_t are star-shaped with respect to O , and every solution G is such an F_t .*

The proofs are similar to the proofs of the foregoing propositions.

More generally, we can say: let the absolutely elliptic differential equation $\Phi = 0$ have a one-parameter family of compact solutions F_t , $t > 0$, F_t continuously dependent on t , $F_t \rightarrow O$ for $t \rightarrow 0$, $d(O, F_t) \rightarrow \infty$ for $t \rightarrow \infty$ ($d(X, Y)$ is the distance between X and Y), and all F_t are supposed to be without self-intersections, $O \in I_t$ for all $t > 0$ (I_t = interior of F_t); the positive normal is taken as above; then every solution G (with the properties given above) is such an F_t .

Added in proof. Since submitting this paper, I discovered that A. D. Alexandrov got theorems similar to the propositions above in *Uniqueness theorems for surfaces in the large*, Vestnik Leningrad University, 1958, no. 7, pp. 14–26 (Russian).

¹⁰ a_v of class C^1 , a_v may depend on the normal of F . Then we assume that a_v is a function of class C^1 on the tangent bundle of $R^{n+1} \setminus S$ (extended homothetically to the tangent bundle of $R^{n+1} - O$). The same remark applies to f (also in the previous propositions).

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SOME RESULTS ON TAME DISKS AND SPHERES IN E^3

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It is well known that there are cells and spheres of dimensions one and two in Euclidean 3-space E^3 which fail to be locally tame at a single point. Artin and Fox [1] have given many examples. Several authors have given characterizations, such as [11; 12 and 13], of the tame cells among the “almost polyhedral” cells. The results in this paper are of the same general nature.

We use the term “disk” to mean a topological closed 2-cell. If D is a disk, then D^0 denotes the interior of D and BdD denotes the boundary of D . The same notation also will be applied to an arc (a topological closed 1-cell).

The Theorem 0 below is stated without proof here, since a complete proof appears in Lemma 5.1 of [7]. We have used this result extensively and have noticed many modifications of it in the literature. It therefore seems useful to state it explicitly.

THEOREM 0. *Let D be a disk in E^3 which is locally polyhedral except at points of BdD . If U is any open set containing D^0 , then there is a 3-cell C^3 in $U \cup BdD$ such that (i) $D \subset C^3$, (ii) D spans BdC^3 , (iii) and C^3 is locally polyhedral except on BdD .*

THEOREM 1. *Let D be a disk in E^3 and let A be a tame arc on D spanning BdD . If D is locally polyhedral except on A , then D is tame.*

PROOF. A remark will suffice to show that D is locally tame at each point of D^0 . In [4] it is shown that each point x in A^0 lies in the