A TOTALLY ORDERED INTEGRAL DOMAIN WITH A CONVEX LEFT IDEAL WHICH IS NOT AN IDEAL

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The question of the existence of such a ring was raised by Donald Johnson.¹ It suffices to find a cancellative totally ordered semigroup which has a lower segment I that is a left ideal but not an ideal. For suppose that S is such a semigroup and let R be the semigroup ring of S over the integers Z . Then R can be ordered lexicographically (first by S and then by Z) and it is well known that R is an ordered integral domain. Let $\mathcal{J}$ be the set of all elements of R of the form $c_1x^{\gamma_1} + \cdots + c_nx^{\gamma_n}$, where $\gamma_1, \dots, \gamma_n \in I$. Since I is a lower segment of S , $\mathcal{J}$ is a convex subgroup of R and it is easy to verify that $\mathcal{J}$ is a left ideal but not an ideal.

We now construct such a semigroup. The following ordered group, from which we shall extract S , is due to Conrad [1]. Let H be the small direct sum of the integers over the integers ordered lexicographically. Define σ from the integers to the ordered automorphism group of H in the following manner: for $a \in H$ we use a_j to denote the j th component of a ; then let $(a\sigma(i))_j = a_{j-i}$. Now let $G = Z \times H$ ordered lexicographically first by Z then by H , where $(i, a) + (j, b) = (i+j, a\sigma(j) + b)$. Then G is an ordered group. Next let

$$S = \{(i, a) \in G \mid i \leq 0, a_j \geq 0 \text{ for all } j, \text{ and } a_j = 0 \text{ for } j > 0\}.$$

It is easily seen that S is a subsemigroup of G . Clearly S is cancellative and contains the identity $(0, 0)$ of G . Let

$$I = \{(i, a) \in S \mid \text{either } i = -1 \text{ and } a_0 = 0, \text{ or } i < -1\}.$$

Then I is a lower segment of S which is a left ideal. But I is not a right ideal because $(-1, 0) \in I$ and if $(0, b) \in S$ such that $b_0 > 0$ then $(-1, 0) + (0, b) = (-1, b) \notin I$.

REFERENCE

1. P. Conrad, *Extensions of ordered groups*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 516-528.

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¹ After this paper was submitted for publication the author learned that Johnson found a different solution to the problem, which will be published.