ON THE REALIZABILITY OF SINGULAR COHOMOLOGY GROUPS[1]

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Let $H_n(X)$ and $H^n(X)$ be the integral singular homology and cohomology groups of a space X and let $\mathcal{G}$ be the category of abelian groups. Then it is well known that for every sequence $(A_1, A_2, \cdots, A_n, \cdots)$ with the $A_n \in \mathcal{G}$, there exists a space X such that $H_n(X) \approx A_n$ for all $n > 0$. We will show that the analogous statement for cohomology is false. In fact we prove:

THEOREM. *There exists no space X and integer $n \geqq 1$ such that $H^{n-1}(X) = 0$ and $H^n(X) \approx Z_0$ (the additive group of the rationals).*

In the proof the following results will be used.

(a) Z_0 has no nontrivial direct sum decomposition (trivial).

(b) $\mathrm{Hom}(A, Z)$ is not divisible for any $A \in \mathcal{G}$ (trivial).

(c)[2] Let $A \in \mathcal{G}$ and $\mathrm{Hom}(A, Z) = 0$. Then $\mathrm{Ext}(A, Z)$ is divisible if and only if A is torsionfree and $\mathrm{Ext}(A, Z)$ is torsionfree if and only if A is divisible.

PROOF. We will write $\mathrm{Hom}\, B$ and $\mathrm{Ext}\, B$ instead of $\mathrm{Hom}(B, Z)$ and $\mathrm{Ext}(B, Z)$. For any integer $m > 1$ consider the exact sequence

$$0 \to {}_mA \to A \xrightarrow{m} A \to A_m \to 0.$$

Because $\mathrm{Hom}\, A = \mathrm{Hom}\, {}_mA = 0$ application of the functor Ext yields the exact sequence

$$0 \to \mathrm{Ext}\, A_m \to \mathrm{Ext}\, A \xrightarrow{m} \mathrm{Ext}\, A \to \mathrm{Ext}\, {}_mA \to 0$$

and hence $\mathrm{Ext}\, A_m = {}_m(\mathrm{Ext}\, A)$ and $\mathrm{Ext}\, {}_mA = (\mathrm{Ext}\, A)_m$. For any torsion group T, $\mathrm{Ext}\, T = 0$ if and only if $T = 0$. Hence ${}_mA = 0$ if and only if $\mathrm{Ext}\, {}_mA = (\mathrm{Ext}\, A)_m = 0$ and $A_m = 0$ if and only if $\mathrm{Ext}\, A_m = {}_m(\mathrm{Ext}\, A) = 0$. The proposition now follows from the fact that a group $B \in \mathcal{G}$ is torsionfree if and only if ${}_mB = 0$ for all $m > 1$ and that B is divisible if and only if $B_m = 0$ for all $m > 1$.

(d) If $A \in \mathcal{G}$ is torsionfree and divisible, then A is a vector space over Z_0 (trivial).

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[2] The first half of this proposition was proved by R. J. Nunke (Illinois J. Math. vol. 3 (1959) p. 230) without the restriction $\mathrm{Hom}(A, Z) = 0$.

(e) If $j: A \to B \in \mathcal{G}$ is a monomorphism, then $\operatorname{Ext}(j, Z): \operatorname{Ext}(B, Z) \to \operatorname{Ext}(A, Z)$ is an epimorphism (trivial).

(f) $\operatorname{Ext}(Z_0, Z)$ is not countable.

PROOF. The exact sequence $0 \to Z \to Z_0 \to Z_0/Z \to 0$ induces an exact sequence

$$0 \to \operatorname{Hom}(Z_0, Z_0) \to \operatorname{Hom}(Z_0, Z_0/Z) \to \operatorname{Ext}(Z_0, Z) \to 0.$$

As $\operatorname{Hom}(Z_0, Z_0) \approx Z_0$ is countable it suffices to show that $\operatorname{Hom}(Z_0, Z_0/Z)$ is not. For every sequence $a_1, a_2, \cdots, a_n, \cdots \in Z_0/Z$ such that $na_n = a_{n-1}$ for all n there clearly is a homomorphism $f: Z_0 \to Z_0/Z$ such that $f(1/n!) = a_n$. As the set of these sequences is not countable neither is $\operatorname{Hom}(Z_0, Z_0/Z)$.

PROOF OF THE THEOREM. Let X be a space such that $H^{n-1}(X) = 0$ and $H^n(X) \approx Z_0$. By the universal coefficient theorem

$$\begin{aligned} 0 = H^{n-1}(X) &\approx \operatorname{Hom}(H_{n-1}(X), Z) + \operatorname{Ext}(H_{n-2}(X), Z) \\ Z_0 \approx H^n(X) &\approx \operatorname{Hom}(H_n(X), Z) + \operatorname{Ext}(H_{n-1}(X), Z). \end{aligned}$$

Hence $\operatorname{Hom}(H_{n-1}(X), Z) = 0$, by (a) and (b) $Z_0 \approx \operatorname{Ext}(H_{n-1}(X), Z)$ and thus, by (c) $H_{n-1}(X)$ is torsionfree and divisible. But then (d), (e) and (f) imply that $\operatorname{Ext}(H_{n-1}(X), Z)$ is not countable which is a contradiction, q.e.d.

REMARK. It is not known whether in the theorem the hypothesis $H^{n-1}(X) = 0$ can be omitted, i.e., whether Z_0 can be a singular cohomology group at all.

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