RADICAL EXTENSIONS OF RINGS

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Jacobson's generalization [5, Theorem 8] of Wedderburn's theorem [8] states that an algebraic division algebra over a finite field is commutative. These algebras have the property that some power² of each element lies in the center. Kaplansky observed in [7] that any division ring, or, more generally, any semisimple ring, in which some power of each element lies in the center is commutative. Kaplansky's idea was generalized in [1], and radical extensions of arbitrary subrings were studied, where the ring A is a *radical extension* of the ring B in case each $a \in A$ is *radical* over B in the sense that some power of a lies in B . In this connection Theorem A of [1] states:

If A is a simple ring with a minimal one-sided ideal, and radical over a subring $B \neq A$, then A is a field.

This is the best possible result of this type for which A/B is radical, and no restriction is placed on B (best in the sense that there exist noncommutative primitive rings with minimal one-sided ideals and radical over proper simple subrings [1, §4]). The starting point for the present investigation is the observation that if A is a division ring and radical over center, then A is a radical extension of both a division subring, and a commutative subring. Accordingly, the present paper is devoted to the study of rings A which are radical over (1) division subrings, or (2) commutative subrings. In connection with (1), there exists the following generalization of the Wedderburn-Jacobson-Kaplansky theorems on division rings.

THEOREM 1.1. *If A is a ring with no nil ideals $\neq \{0\}$, and if A is radical over a division subring $B \neq A$, then A is a field.*

Corresponding to (2), one has the following extension of Kaplansky's theorem for semisimple rings. In it (and throughout this paper) $J(A)$ denotes the *Jacobson radical* of the ring A .

THEOREM 1.2. *If A is radical over a commutative subring B , and if $J(A) = \{0\}$, then A is commutative.*

Kaplansky's theorem [7] is, then, the special case of Theorem 1.2 for which B is central.

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² Here, and below, the "zero power" is excluded.

In §2 necessary and sufficient conditions are given in order that a ring A be a radical extension of a division ring B . These conditions are strong enough to enable one to deduce that the only radical extensions of a division ring B , when B is either noncommutative, or has characteristic 0, are the trivial ones: $B \oplus N$, where N is an arbitrary nil ring.

In §3 under the hypotheses (i) $J(A) \neq A$ and (ii) A has no nil ideals $\neq \{0\}$, it is shown that in order that A be a radical extension of a simple subring B it is necessary that A be a primitive ring. Then, if B contains an identity, A itself is simple. Moreover, in order that A be primitive (resp. semisimple) and a radical extension of a ring B , it is necessary that B itself be primitive (resp. semisimple; cf. Theorem 3.2 ff).

§4 contains an application of Theorem 1.1 on the structure of the multiplicative group of a division ring.

1. This section contains proofs of Theorems 1.1 and 1.2, and several results related to theorems of Herstein.

PROOF OF THEOREM 1.1. Since A is not a radical ring in the sense of Jacobson [6], that is, since the Jacobson radical $J(A) \neq A$, A contains a primitive ideal P . If $P \cap B \neq \{0\}$, then $P \supseteq B$ and $A - P$ would be a nil primitive ring. This contradiction shows that $P \cap B = \{0\}$. Then P , being a radical extension of $\{0\} = P \cap B$, is nil. By hypothesis, $P = \{0\}$, so that A is a primitive ring. Similarly, since A contains no nil left ideals, if Q is any nonzero left ideal in A , necessarily $Q \supseteq B$. In particular, B is contained in the intersection of the nonzero modular maximal left ideals of A . Since the intersection of all of the modular maximal left ideals is $J(A)$, and since $J(A) = \{0\}$, it follows that $\{0\}$ is a modular maximal ideal, so that A is a division ring. Then A , being a division ring which is radical over a division subring $B \neq A$, is a field, according to [1, Theorem A].

PROOF OF THEOREM 1.2. The primitive homomorphic images of A inherit the hypotheses on A . Since A is a subdirect sum of primitive rings, and since a subdirect sum of commutative rings is itself commutative, it suffices to consider the case where A is primitive. Assume that A is a noncommutative primitive ring. Then, by Kaplansky's theorem, B is not contained in the center Z of A . By [1, Theorem A], A cannot be a division ring. Thus, A has a representation as a dense ring of linear transformations on a vector space V of dimension > 1 over a division ring. Choose $b \in B$, $b \notin Z$. Then, there exists $x \in V$ such that x and xb are linearly independent. By the density of A , there exists $a \in A$ mapping both x and xb onto x . Then, for this a , no power of a commutes with b . Then, since B is commutative, no

power of a can lie in B . Thus, A/B is not radical, a contradiction which proves the theorem.

Theorem 1.2 has the following corollaries.

COROLLARY 1.3. *If A is a noncommutative semisimple ring, and if $a \in A$ is noncentral, then there exists $b \in A$ such that $b^n a \neq ab^n$, for all $n \geq 1$.*

PROOF. Let $a \in A$ be noncentral, and let a' denote the corresponding element in the primitive homomorphic image A' of A . For some A' , a' is noncentral. Then in this A' there exists, by the proof of the theorem, b' such that no power of b' commutes with a' . Then if $b \in A$ maps onto b' under the homomorphism $A \rightarrow A'$, then no power of b commutes with a .

COROLLARY 1.4. *If A is semisimple, and if N is a fixed natural number such that to each pair $x, y \in A$, x is radical over the centralizer of y^N in A , then A is commutative.*

PROOF. If A were noncommutative, then, by Kaplansky's theorem, there would exist $y \in A$ such that y^N is noncentral. Then A is not radical over the centralizer of y^N by the corollary. This contradiction completes the proof.

In connection with the last corollary, it is of interest to ask what conditions on A imply commutativity, if A satisfies:

(*) *For every $x, y \in A$, there exist natural numbers n, m such that $x^n y^m = y^m x^n$.*

Since any ring A radical over a commutative subring B satisfies (*), in view of Theorem 1.2 it is natural to ask if every semisimple ring satisfying (*) is commutative. The next theorem gives a partial answer.

THEOREM 1.5. *If A is a primitive ring with a minimal left ideal, or if A is a subdirect sum of such rings, and if A satisfies (*), then A is commutative.*

PROOF. It suffices to assume that A is primitive, and even further that A coincides with its socle, that is, that A is simple with a minimal left ideal. If $y \in A$, then

$$(y)' \subseteq (y^2)' \subseteq (y^6)' \subseteq \cdots \subseteq (y^{n!})' \subseteq \cdots,$$

where as before the primes denote centralizers in A . Clearly (*) implies that A is radical over $B_y = \bigcup_1^\infty (y^{n!})'$, so by [1, Theorem A] noncommutativity of A implies equality $A = B_y$ for each $y \in A$. Thus, if

$x \in A$, then for each $y \in A$, $x \in (y^m)'$, that is, $y^m \in x'$, for suitable m , so that A is radical over x' for each $x \in A$. Then by Corollary 1.4 commutativity of A follows, a contradiction which proves the theorem. Herstein's extension [3] of Kaplansky's theorem states that a ring A containing no nil ideals $\neq \{0\}$, and radical over center is commutative. This raises the following question which is left open:

If A is a ring with no nil ideals $\neq \{0\}$ and radical over a commutative (possibly semisimple) subring B , is A necessarily commutative?

If B is a semisimple ideal, the answer is yes. This can be seen as follows: $J(B) = \{0\}$ so that $J(A) \cap B = \{0\}$, whence $J(A)$ is nil. By hypothesis, $J(A) = \{0\}$, and Theorem 1.2 applies.

In view of Herstein's [4, Theorem 1] one might ask if any ring A with no nil ideals $\neq \{0\}$, and satisfying (*) is commutative. The question is open even if m is fixed as in Corollary 1.4.

2. The main purpose of this section is to present a characterization of non-semisimple rings which are radical over division rings. The structure theory for such rings can be reduced to the ones which are directly irreducible. An example of a radical extension of a division ring which is noncommutative, non-semisimple, and directly irreducible is given.

LEMMA 2.1. *Let A be a ring with a subring B such that B contains an identity e , and such that A/B is radical. Then*

$$A = N \oplus Q,$$

where $N = (1-e)A(1-e)$ is a nil ideal, and $Q = eAe$.

PROOF. Write $A = eAe + fAf + eAf + fAe$, where (formally) $f = 1 - e$, and let $u \in eAf$. Since $(e+u)^n = e+u$, for all n , $e+u \in B \subseteq eAe$. Then, $0 = u \in eAe \cap eAf = \{0\}$, that is, $eAf = \{0\}$. By symmetry, $fAe = \{0\}$, so that $A = eAe \oplus fAf$. Clearly, fAf , being radical over $eAe \cap fAf = \{0\}$, is nil.

THEOREM 2.2. *In order that a ring Q be a radical extension of a subring B , the following three conditions are sufficient:*

- (α) Q contains a nil ideal I such that Q is radical over $B+I$;
- (β) for all $i \in I$, B is radical over $B \cap i'$, where i' is the centralizer of i in Q ;
- (γ) Q has prime characteristic p .

PROOF. Let $a \in Q$, and, by (α), choose m such that $a^m \in B+I$, $a^m = b+i$, $b \in B$, $i \in I$. By (β), choose N such that $b^N i = i b^N$, and write $a^m = b^N + j$, where $j \in I$. Since b^N commutes with a^m , b^N commutes

with a^{mN} , so that b^N commutes with $j = a^{mN} - b^N$. Then if $p' \geq$ the index of j ,

$$a^{mNp'} = b^{Np'} \in B$$

as desired.

THEOREM 2.3. *The radical extensions of a division ring B are the direct sums*

$$Q \oplus N,$$

where N is an arbitrary nil ring, and Q is a directly irreducible radical extension of B . Necessary and sufficient conditions for a directly irreducible ring $Q \supset B$ (and $Q \neq B$) to be a radical extension of B are as follows:

- (1) The identity e of B is the identity of Q ; $J = J(Q)$ is a nil ideal; and $Q - J$ is a field which is radical over $(B + J) - J$.
- (2) B is a field, and for all $q \in B + J$, Q is radical over $B \cap q'$, where q' is the centralizer in Q of q .
- (3) Q has prime characteristic.³

PROOF. If A is a radical extension of B , then by Lemma 2.1, $A = Q \oplus N$, where $Q = eAe$, and $N = (1 - e)A(1 - e)$ is nil. Suppose that $Q = L \oplus K$, where L and K are ideals of Q . Both $L/(L \cap B)$ and $K/(K \cap B)$ are radical extensions. Since Q is non-nil, it can be assumed that $L \cap B \neq \{0\}$. Then, since $B \subseteq L$, $e \in L$, so that $L = Q$ and $K = \{0\}$, that is, Q is directly irreducible. Conversely, $Q \oplus N$, Q and N as above, are radical extensions of B .

(1) From now on, assume that $Q \neq B$. Since $J \cap B = \{0\}$, J is nil; $Q - J$ is a field by Theorem 1.1, if $Q \neq B + J$. If $Q = B + J$, that $Q - J$ is a field follows from (2), so that the proof of (1) is reduced to that of (2). Furthermore, if $J = \{0\}$, then Q is a field by Theorem 1.1, since then Q is a ring with no nil ideals $\neq \{0\}$, and is a proper radical extension of B . Thus, in the case $J = \{0\}$, (2) is trivial, and (3) follows from Kaplansky's lemma [7]. Hence, only the $J \neq \{0\}$ case of (2) and (3) remains.

(2) The verbatim proof of [1, Lemma 1] yields the following result which is needed below.⁴ (The notation is that of [1].)

Let D be a ring with identity 1, and let Δ be a division subring $\neq D$ such that D/Δ is radical. Let $v \in D$, $v \notin \Delta$ be such that $(v + c_i)^{-1}$ exists

³ Considerably more information can be squeezed out of the hypotheses. See, for example, a following article of mine [2], and Kaplansky's lemma [7].

⁴ This result is extensively generalized in [2] in a situation where arbitrary polynomials play the role of the nomomials x^n . Nevertheless, even the proof of this more general result is little more than a rewriting of the proof of [1, Lemma 1].

for different $c_i \in \Delta$, $i=1$ and 2 . Then, if $d \in \Delta$ commutes with both c_1 and c_2 , then $d^n v = v d^n$, for some $n \geq 1$.

Choose $0 \neq j \in J$, and $b \in B$. Since $(j+c)^{-1}$ exists for all (more than one) nonzero $c \in B \cap b'$, by the result just announced, b is radical over j' . By setting $v = j+d$, $0 \neq j \in J$, $0 \neq d \in B$, it follows in the same way that b is radical over $(j+d)'$. These two statements together show that $B/(B \cap q')$ is radical, for all $q = j+d \in J+B$. Since Q/B is radical, so is $Q/(B \cap q')$. If $B = B \cap b'$ for all $b \in B$, then B is commutative. Otherwise, B is radical over a proper subring, and B is commutative by [1, Theorem A or B]. This completes the proof of (2).

(3) Let $0 \neq j \in J$ have index of nilpotency $m \geq 2$, and let n be the minimal natural number for which

$$(e+j)^n = e + nj + \cdots + j^n \in B.$$

Then,

$$b = nj + \cdots + j^n \in B \cap J = \{0\}.$$

If $m=2$, then $b = nj = 0$. If $m > 2$, then $b j^{m-2} = n j^{m-1} = 0$. Hence, in either case, $n j^{m-1} = 0$, showing that $n e = 0$. Since $n \neq 0$, B has prime characteristic which divides n . Then $Q = e Q e$ also has prime characteristic.

The sufficiency of (1)–(3) follows from Theorem 2.2.

REMARK. Theorem 1.1 is contained in the statement of Theorem 2.3.

The following corollary is immediate.

COROLLARY 2.4. *If B is a division ring which is noncommutative, or has characteristic 0, then the radical extensions of B are the direct sums $B \oplus N$, where N is an arbitrary nil ring.*

EXAMPLE. Below is an example illustrating Theorem 2.3 in the case $J(Q) \neq \{0\}$ and $Q = e A e$ is directly irreducible (as before, e is the identity of B). Let Q be the algebra of four dimensions over $GF(2)$ with a basis $e, \alpha, j, \alpha j$, and the table:

	e	α	j	αj
e	e	α	j	αj
α	α	$\alpha - e$	αj	$(\alpha - e)j$
j	j	$(\alpha - e)j$	0	0
αj	αj	$(\alpha - e)j$	0	0

One easily verifies: (1) Q is associative; (2) $J(Q) = \{0, j\}$; and (3) $B = (0, e, \alpha, \alpha - e) \cong GF(4)$. Since Q satisfies the conditions of Theorem 2.2, Q/B is radical. Also one may verify this directly, since if $x \in Q$, then $x = a + bj$, where $a, b \in B$, and

$$x^6 = [(a + bj)^3]^2 = [e + i]^2 = e \in B,$$

where $a \neq 0$ (where $i \in J$), and $x^2 = 0$, when $a = 0$. (Note that $B \cong GF(4)$ is radical over $GF(2)$, a fact predicted by Theorem 2.3, since J does not lie in the centralizer of B in Q .)

3. As I mentioned in the introduction, [1] contains examples of noncommutative, nonsimple primitive rings (of arbitrary characteristic) which are radical over their socles, that is, over simple subrings, a fact which precludes the existence of theorems of type 1.1 for radical extensions of simple rings. Nevertheless, simplicity of B has noteworthy implications for radical extensions of B , as the following theorem indicates.

THEOREM 3.1. *If A is a ring with no nil ideals $\neq \{0\}$, and if A is a radical extension of a simple ring B , A not a radical ring, then A is a primitive ring.*

PROOF. If I is any nonzero ideal of A , since I is non-nil, then $I \cap B \neq \{0\}$, so that $I \supseteq B$, by simplicity of B . Now $B \not\subseteq J(A)$, since, otherwise, $A - J(A)$ would be a nil ring, and, then, $A = J(A)$, contrary to the hypothesis. Hence, $J(A) = \{0\}$, and A is primitive in the same way as the proof of Theorem 1.1.

COROLLARY. *If A is a ring with no nil ideals $\neq \{0\}$, if A is a radical extension of a simple ring B , and if B contains an identity e , then A is a simple ring with identity e .⁵*

PROOF. Clearly, A is not a radical ring, so that A is primitive by the theorem. By Lemma 2.1, $A = eAe$, so that e is an identity element in A . The first line in the proof of the theorem shows: If I is any nonzero ideal of A , then $I \supseteq B$. Then $e \in I$, so that $I = A$ is simple.

Now let A be a noncommutative primitive ring, and let B be a subring such that A/B is radical. Represent A as a dense ring of l.t.'s in a vector space V over a division ring D . Since A is noncom-

⁵ It is not known whether there exists a noncommutative simple ring which is radical over a subring B (even if one assumes that B is simple, and that A contains an identity element e). As I pointed out in [1], a proof of the nonexistence of such rings (for which the parenthetical hypotheses are omitted) will preclude the existence of non-nilpotent simple nil rings, since these are necessarily noncommutative and radical over every subring.

mutative, A is not a division ring, and so $\dim V > 1$. I wish to show that B is isomorphic to a dense ring of l.t.'s in V . It suffices to show, if V_n is vector subspace of V of finite dimension n , that every l.t. of V_n induced by an element of A can be induced by an element of B , $n = 1, 2, \dots$. (Then, every l.t. in V_n can be induced by an element of B .) Accordingly, let $U = \{a \in A \mid V_n a \subseteq V_n\}$, and $K = \{a \in A \mid V_n a = \{0\}\}$. By the density of A , if $\bar{U}$ is the difference ring $U - K$, then $\bar{U} \cong D_n$, the complete ring of $n \times n$ matrices over D . Since U is radical over $Q = U \cap B$, then $\bar{U}$ is radical over $\bar{Q}$. First suppose $n > 1$. Then, by [1, Theorem A], $\bar{U} = \bar{Q}$, that is, $U = Q + K$. This shows, if $\bar{a}$ is any l.t. in V_n which is induced by a in A , then $a \in U$ so that $a = b + k$, with $b \in Q \subseteq B$, $k \in K$. Then $b \in B$ induces $\bar{a}$ as well. The case $n = 1$ remains. Now V_1 is contained in a 2-dimensional subspace V_2 , and if $\bar{a}_1$ is any l.t. in V_1 , there exists a l.t. $\bar{a}_2$ in V_2 which induces $\bar{a}_1$. Then, if $b \in B$ induces $\bar{a}_2$, then b also induces $\bar{a}_1$. Thus, in all cases, the l.t.'s in V_n can be induced by elements of B . This completes the proof of the theorem below.

THEOREM 3.2. *Let A be a noncommutative primitive ring, and let B be a subring such that A/B is radical. Then, if A is represented as a dense ring of l.t.'s in a vector space V over a division ring, then B is isomorphic to a dense subring of l.t.'s in V .*

COROLLARY. *If A is a primitive ring, and if B is a subring such that A/B is radical, then B is a primitive ring.*

PROOF. If A is noncommutative, then B is primitive by the theorem. If A is commutative, then A is a field, and so is B , by [1, Lemma 3].

Note that Theorem 3.2 (together with [1, Theorem A]) is strong enough to imply both Theorems 1.1 and 1.2, after the reduction to the primitive case.

The following was suggested by a remark of the referee:

If A is a semisimple ring, and if A is radical over a subring B , then B is semisimple.

PROOF. Let $\mathcal{O}$ denote the set of all primitive ideals of A . Then $\{P \cap B \mid P \in \mathcal{O}\}$ is a collection of primitive ideals of B . This can be seen as follows: Since $A - P$ is a primitive ring which is radical over the subring $(P + B) - P$, by the corollary, $(P + B) - P$ is a primitive ring, for each $P \in \mathcal{O}$. Now $B - (P \cap B)$, being isomorphic to $(P + B) - P$, is also a primitive ring, that is, $(P \cap B)$ is a primitive ideal of B , for each $P \in \mathcal{O}$. Since $\bigcap_{P \in \mathcal{O}} P = \{0\}$, clearly, $\bigcap_{P \in \mathcal{O}} (P \cap B) = \{0\}$, and, therefore, B is semisimple.

4. This section contains an application of Theorem 1.1. Let A be a division ring with center $Z \neq A$; let B be a division subring, $B \neq A$, $B \not\subseteq Z$; let $S^* = \{0 \neq a \in S\}$ and $N(S) = \{a \in A^* \mid a^{-1}Sa = S\}$, for any division subring S of A ; and let

$$N_0 = \bigcap_{a \in A^*} N(a^{-1}Ba).$$

Hattori has shown that $N_0 = Z^*$ under the hypothesis that A is finite dimensional over Z , and $A \supset B \supset Z$. This result, together with Kaplansky's theorem [7], shows that A^*/N_0 is not a torsion group, when $[A:Z] < \aleph_0$. The principal result of my article, *On conjugates in division rings*, Canad. J. Math. vol. 10 (1958) pp. 374–380, is the following theorem that holds for arbitrary A and B as above: A^*/N_0 is not finite (in fact, $N(B)$ has infinite index in A^*). Whether A^*/N_0 can be torsion for some (necessarily infinite dimensional) A is an open question. If A^*/N_0 is torsion, then Theorem 1.1 shows that the division subring generated by N_0 coincides with A . Aside from this, Theorem 1.1 can be used to obtain the following

THEOREM 4.1. *Assume the notation directly above. If A has characteristic $p > 0$, then A^*/N_0 is not a p -group.*

PROOF. Assume that A^*/N_0 is a p -group. Then, if $0, -1 \neq v \in A$, there exists an exponent $e > 0$ such that both v^{p^e} and $(v+1)^{p^e} \in N$, $N = N(B)$. Then, since both v^{p^e} and $v^{p^e} + 1$ lie in N , by Brauer's result (cf., loc. cit., Proposition 3), it follows that:

(*) *To each $v \in A$ there corresponds $e > 0$ such that $v^{p^e} \in B \cup B'$ (set union).*

(Here, B' denotes the centralizer of B in A .) This, together with Theorem 1.1, implies that $B \not\subseteq B'$ and that $B' \not\subseteq B$, since a noncommutative division ring cannot be radical over a proper subring. Since B is thereby not equal to $B \cap B'$, B is itself noncommutative, so that, by applying Theorem 1.1 again, there exists an element $a \in B$ which is not radical over $B \cap B'$. Let $v = a + b$, where $b \in B'$. By (*), $v^{p^e} = a^{p^e} + b^{p^e} \in B \cup B'$ for some $e > 0$. Since $a^{p^e} \notin B'$, necessarily $v^{p^e} \in B$, whence $b^{p^e} \in B \cap B'$. This shows:

(**) *Some power p^e of each $b \in B'$ lies in B .*

Now let v be any element of A not radical over B . By (*), $v^{p^e} \in B'$ for some $e > 0$. By (**), $v^{p^k} \in B$ for some $k \geq e$, contrary to the choice of v . This contradiction establishes the theorem.

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NOTE ON THE NONVANISHING OF $L(1)$

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It is well known that if $\chi(m)$ is a real nonprincipal character (mod k), then

$$L(1) = \sum_1^{\infty} \frac{\chi(m)}{m} \neq 0,$$

and many proofs have been found. We give a very simple proof when $k=p$ an odd prime, in which case $\chi(m) = (m/p)$, the Legendre symbol. This makes it possible to simplify the proof that if $p \nmid a$, then there are infinitely many primes congruent to a modulo p . Write

$$\zeta = e^{2\pi i/p}, \quad P = \frac{\prod_n (1 - \zeta^n)}{\prod_r (1 - \zeta^r)},$$

where n runs through the quadratic nonresidues of p and r runs

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