EXTREME POINTS OF POLAR CONVEX SETS

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1. Introduction. Suppose that E is a normed linear space and that $f \in E^*$, $\|f\| \leq 1$. There is a closed, convex subset E_f of E , intimately related to f , which is defined as follows: $E_f = \{x \in E: \|x\| - f(x) \leq 1\}$. R. C. Buck [2] has proved (among other things) that f is an extreme point of the unit ball U^* of E^* if and only if $E = E_f - E_f$, i.e., f cannot be expressed as the midpoint of a line segment in U^* if and only if every point of E is of the form $u - v$, where u and v are in E_f . Consider this result from a different viewpoint: Letting

$$C = \{x \in E: \|x\| \leq 1\},$$

the theorem gives, in terms of f and the norm in E (i.e., the gauge functional of C), a necessary and sufficient condition that an element f in the polar C^0 of C in E^* be an extreme point of C^0 . Note that the set C is closed, convex, symmetric about the origin, and has nonempty interior. The main purpose of the present note is to give an elementary proof of a similar characterization of the extreme points of C^0 without making any assumptions about C other than it be a convex subset, containing the origin, of some linear space F . In order to consider the polar of C we naturally require that F be in duality with another space G ; the duality inherent in our formulation will enable us to deduce a dual version of Buck's theorem.

2. The main theorems. Using the notation of [1], let F and G be real or complex linear spaces in duality under $\langle, \rangle$. (In all that follows, the symbol $\langle, \rangle$ will stand for $\text{Re}\langle, \rangle$ in case F and G are over the complex field.) Let C be a convex subset of F which contains the origin ϕ , and let p be the gauge functional defined by C ; for $x \in F$, $p(x) = \inf\{\lambda > 0: \lambda^{-1}x \in C\}$. Note that $0 \leq p(x) \leq \infty$ (the latter equality holding if and only if $\lambda^{-1}x \notin C$ for all $\lambda > 0$), $p(x+y) \leq p(x) + p(y)$ for all x, y in F , $p(\alpha x) = \alpha p(x)$ if $\alpha > 0$, and $p(x) \leq 1$ if $x \in C$. Let C^0 be the polar of C in G , i.e., $C^0 = \{y \in G: \langle x, y \rangle \leq 1 \text{ for all } x \in C\}$. It follows from the definitions that if $y \in C^0$ and $x \in F$, then $\langle x, y \rangle \leq p(x)$. Supposing $y \in C^0$, let $F_y = \{x \in F: p(x) - \langle x, y \rangle \leq 1\}$; the subadditivity of p implies that F_y is convex. The topology $\sigma(F, G)$ on F is locally convex, and hence [3, p. 22] a convex subset K of F is not $\sigma(F, G)$ -dense

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in F if and only if there exists a $z \in G \sim \{\phi\}$ and $b > 0$ such that $\langle x, z \rangle \leq b$ for all x in K . By considering $b^{-1}z$ instead of z , we may, if we wish, assume $b = 1$.

THEOREM 1. *A point $y \in C^0$ is an extreme point of C^0 if and only if $F_y - F_y$ is $\sigma(F, G)$ -dense in F .*

PROOF. Since $F_y - F_y$ is convex, if it is not $\sigma(F, G)$ -dense in F , there exists a $z \in G \sim \{\phi\}$ as above. Since $x \in F_y - F_y$ if and only if $-x \in F_y - F_y$, and since $F_y \subset F_y - F_y$, we have $|\langle x, z \rangle| \leq 1$ for all $x \in F_y$. We will show that the points $y \pm z$ are in C^0 ; since $y = 2^{-1}(y+z) + 2^{-1}(y-z)$, we can then conclude that y is not an extreme point of C^0 . We must show, then, that $\langle x, y \pm z \rangle \leq 1$ for all $x \in C$. Take $x \in C$ (so $p(x) \leq 1 < \infty$) and let $\alpha = p(x) - \langle x, y \rangle$. If $\alpha = 0$, then $\lambda x \in F_y$ for all $\lambda > 0$; since $|\langle x, z \rangle| \leq \lambda^{-1}$, we have $\langle x, z \rangle = 0$ and hence $\langle x, y \pm z \rangle = \langle x, y \rangle \leq 1$. If $\alpha > 0$, then $1 = \alpha^{-1}\alpha = p(\alpha^{-1}x) - \langle \alpha^{-1}x, y \rangle$, so $\alpha^{-1}x \in F_y$. Hence $\langle \alpha^{-1}x, z \rangle \leq 1$ or $\pm \langle x, z \rangle \leq \alpha = p(x) - \langle x, y \rangle$ and therefore $\langle x, y \pm z \rangle \leq p(x) \leq 1$. Thus, $y \pm z \in C^0$ and we have finished one part of the proof. To prove the converse, suppose that y is not an extreme point of C^0 . There exist u and v in C^0 such that $y = 2^{-1}(u+v)$; equivalently, there exists $z \in G \sim \{\phi\}$ such that $y \pm z \in C^0$ (take $z = v - y$, say). Thus, if $x \in F$, we have $\langle x, y \pm z \rangle \leq p(x)$ or $\langle x, z \rangle \leq p(x) - \langle x, y \rangle$. Therefore $\langle x, z \rangle \leq 1$ for all $x \in F_y$, and hence $\langle x, z \rangle \leq 2$ for all $x \in F_y - F_y$, which shows that the latter is not $\sigma(F, G)$ -dense in F and completes the proof.

Following Buck's example, we will now use Theorem 1 to prove a more general result. If M is a linear subspace of F (not necessarily closed) let $M^\perp = \{y \in G: \langle x, y \rangle = 0 \text{ for all } x \in M\}$ (this is simply M^0). With C as before, we can characterize the extreme points of $C^0 \cap M^\perp$ as follows:

THEOREM 2. *A point $y \in C^0 \cap M^\perp$ is an extreme point of $C^0 \cap M^\perp$ if and only if the set $F_y - F_y + M$ is $\sigma(F, G)$ -dense in F .*

PROOF. Let $C_1 = C + M$ and define p_1 , C_1^0 and F_y^1 (for $y \in C^0$) in the obvious way. By Theorem 1, y is an extreme point of C_1^0 if and only if $F_y^1 - F_y^1$ is $\sigma(F, G)$ -dense in F . The proof is completed by showing that $C_1^0 = C^0 \cap M^\perp$, and that for all y in C_1^0 , $F_y - F_y + M$ is a $\sigma(F, G)$ -dense subset of $F_y^1 - F_y^1$. The former fact follows easily from the definitions. The latter remark may be proved as follows: Note first that for $x \in F$, $p_1(x) = \inf\{p(x+m): m \in M\}$. Since $p(x+m) \leq p(x)$, we have $F_y + M \subset F_y^1$ and hence $F_y - F_y + M \subset F_y^1 - F_y^1$. If $x \in F_y^1$, let $u_n = (1 - n^{-1})x$. Then $p_1(u_n) - \langle u_n, y \rangle < 1$ so there exists $m_n \in M$ such that $p(u_n + m_n) < 1 + \langle u_n, y \rangle = 1 + \langle u_n + m_n, y \rangle$ and therefore $u_n + m_n \in F_y$. Now, if $x - x' \in F_y^1 - F_y^1$, define u_n , u_n' and m_n , m_n' as above;

then $u_n + m_n - (u'_n + m'_n) \in F_y - F_y$ and hence $u_n - u'_n \in F_y - F_y + M$. Since $u_n - u'_n$ clearly converges to $x - x'$ in the $\sigma(F, G)$ topology, this completes the proof.

If E is a topological vector space, then it is in duality with E^* (the space of all continuous linear functionals on E) in the obvious way: $\langle x, f \rangle = f(x)$ for all $x \in E, f \in E^*$. (In keeping with our convention we will assume that $f(x)$ stands for $(\operatorname{Re} f)(x)$ if E is complex.)

THEOREM 3. *Suppose that E is a topological vector space, and that the convex set C contains the origin in its interior. Then, for any subspace M , we have $E = E_f - E_f + M$ if (and only if) f is an extreme point of $C^0 \cap M^\perp$.*

PROOF. By Theorem 2, the "only if" statement is obvious, while, if f is an extreme point of $C^0 \cap M^\perp$, then $E_f - E_f + M$ is $\sigma(E, E^*)$ -dense in E . Let $C' = 2^{-1}[C \cap (-C)]$; then $\phi \in \operatorname{int} C'$, and if $x \in C'$ we have $\pm 2x \in C$ so that $x \in E_f$. Thus, E_f (and therefore $E_f - E_f + M$) has nonempty interior. If there exists $u \in E \sim (E_f - E_f + M)$, then there exists [3, p. 22] a $g \in E^* \sim \{\phi\}$ such that $g(x) \leq g(u)$ for all x in $E_f - E_f + M$, contradicting the $\sigma(E, E^*)$ -density of the latter. Thus, $E = E_f - E_f + M$, which proves the theorem.

It is worth noting that we have not required either C or M to be closed.

3. Applications to normed spaces. Suppose that E is a (real or complex) normed linear space. Denote its unit ball by U , and let U^* denote the unit ball of E^* . The following is an immediate corollary of Theorem 3.

THEOREM 4 (R. C. BUCK). *Let M be a subspace of E and suppose $f \in U^* \cap M^\perp$. Then f is an extreme point of $U^* \cap M^\perp$ if and only if $E = E_f - E_f + M$.*

Note that the above theorem actually characterizes the extreme points of the unit ball of $(E/M)^*$. The following is a dual version of this theorem.

THEOREM 5. *Let M be a closed subspace of E and suppose $x \in U \cap M$. Then x is an extreme point of $U \cap M$ if and only if $E_x^* - E_x^* + M^\perp$ is weak*-dense in E^* .*

PROOF. This follows from Theorem 2 upon recalling that the weak* topology in E^* is the $\sigma(E^*, E)$ topology, and observing that since M is closed, it is the annihilator in E of $M^\perp \subseteq E^*$.

Obviously, the above theorems characterize the extreme points of U^* or U by taking $M = \{\phi\}$ or $M = E$, respectively.

4. Remarks. A natural question arises with respect to Theorem 5 (taking $M=E$): If x is an extreme point of U , is it possible to conclude that $E_x^* - E_x^*$ actually equals E^* ? Letting Q be the natural embedding of E into E^{**} , it is clear that $E_x^* = E_{Qx}^*$, so (by virtue of Theorem 4) this is equivalent to the question: Must the image Qx of an extreme point $x \in U$ be an extreme point of U^{**} ? We will exhibit a normed space for which the answer to this is negative, but the question is still open for *complete* normed (i.e., Banach) spaces. Suppose that E is a normed linear space with unit ball U , and suppose $\hat{E}$ is the completion of E , with unit ball $\hat{U}$. If there exists an $x \in U$ which is an extreme point of U but not of $\hat{U}$, then we see (by using the facts that $E^* = \hat{E}^*$, $E^{**} = \hat{E}^{**}$ and $\overline{QE} = Q(\hat{E})$) that Qx is not an extreme point of $Q\hat{U} = \overline{QU} \subset U^{**}$, and hence is not an extreme point of U^{**} . The problem, then, is to find a space E containing a point x with the above properties. Define a new norm on the (real) space l_2 as follows: The new unit ball $\hat{U}$ in l_2 will be the closed convex hull of the set $\{x \in l_2: x_1 = 0, \sum x_i^2 \leq 1\}$ together with the points $u = (1, 1, 0, 0, 0, \dots)$, $v = (1, -1, 0, 0, \dots)$ and their negatives. Using a theorem of V. L. Klee as in [4, §3], we can construct a dense Hamel basis B for l_2 which contains $x = (1, 0, 0, \dots)$ and $y = (0, 1, 0, 0, \dots)$. Let E be the linear span (not closed) of $B \sim \{y\}$, and let the unit ball U of E be $\hat{U} \cap E$. Then the completion $\hat{E}$ of E is l_2 with the norm induced by $\hat{U}$, and since $x = 2^{-1}(u+v)$, x is not an extreme point of $\hat{U}$. It is an extreme point of U , however, since $y \notin E$ implies that x is the only point of the segment $[u, v]$ which is in E , and (from the definition of $\hat{U}$) any segment in $\hat{U}$ which contains x in its interior must lie in $[u, v]$.

Our Theorem 4 is not formulated in precisely the same manner as Buck's theorem [2, Theorem 2], where it was proved (for E a real space and M closed) in the following form: Let $k > 0$ be an integer and let $E_f^k = \{x \in E: \|x\| - f(x) \leq k^{-1}\}$; then f is an extreme point of $U^* \cap M^\perp$ if and only if $E = E_f^k - E_f^k + M$ for $k = 1, 2, 3, \dots$. This follows from Theorem 4, however, upon observing that $E_f = kE_f^k$ for all $k > 0$.

An elegant reformulation of Theorem 3 (and hence of Theorem 4) may be obtained (as in [2]) by setting $V_f' = \bigcap_1^\infty (k^{-1}E_f - k^{-1}E_f + M)$; since $V_f' = E$ if and only if $E_f - E_f + M = E$, f is an extreme point of $U^* \cap M^\perp$ if and only if the closed subspace V_f' is all of E . This formulation does *not* hold under the minimal hypotheses of Theorem 2, however, since $V_f' = \bigcap_1^\infty (k^{-1}F_f - k^{-1}F_f + M)$ may be a proper subspace of F even though $F_f - F_f + M$ is $\sigma(F, G)$ -dense in F . Indeed, we may take F and G to be E^* and E , respectively, in the example described

above. Letting U^* be C and $M = \{\phi\}$, we have shown that there exists an $x \in C^0 \equiv U$ such that x is an extreme point of C^0 , so $F_x - F_x \equiv E_x^* - E_x^*$ is $\sigma(F, G)$ -dense in F . Since Qx is not an extreme point of U^{**} , however, there exists (as in the proof of Theorem 1) an element $z \in E^{**} \sim \{\phi\}$ which is bounded on $E_x^* - E_x^*$; hence z vanishes on V'_x and therefore V'_x is proper.

Suppose E is a normed space, M a closed subspace of E , and $f \in U^*$. Buck has suggested that (setting $V_f = \bigcap_1^\infty (k^{-1}E_f - k^{-1}E_f)$) $V_f + M$ may always be dense in V'_f . That this need not happen may be seen as follows:

Let E be the space c_0 of real sequences converging to zero, with the supremum norm; then E^* is the space l_1 of absolutely summable sequences. Let f be the functional on E corresponding to the sequence $(2^{-1}, 2^{-2}, 2^{-3}, \dots) \in l_1$, and let M be the closed subspace of c_0 consisting of all x such that $\sum n^{-2}x_n = 0$. We will show that $V_f + M$ is not dense in V'_f . To this end, let (for $n = 1, 2, 3, \dots$) u_n be the sequence in l_1 whose first n terms are zero, and whose k th term is $2^{-(k-1)}$ for $k > n$. Then $\|f \pm u_n\| = \|u_n\| = 1$ and (as in the proof of Theorem 1) each u_n is bounded on $E_f - E_f$ and hence vanishes on V_f . Thus, the elements $2^n(u_n - u_{n+1})$ vanish on V_f ; since (as is easily verified) these latter form a total set over c_0 , we see that $V_f = \{\phi\}$.

Let δ^n be the sequence which is 1 in the n th place and zero elsewhere, and write $2^{n-1}\delta^n = (2^{n-1} \sum_1^n \delta^i) - (2^{n-1} \sum_1^{n-1} \delta^i)$. Thus, for each n , $2^{n-1}\delta^n$ is the difference of two elements from E_f . We will show that the element δ^1 is in $V_f^1 = \bigcap k^{-1}(E_f - E_f + M)$ but not in $M + V_f$. The latter is obvious; we must show, then, that $k\delta^1 \in E_f - E_f + M$ for each $k > 0$. Given k , choose $n > 0$ such that $2^{n-1}n^{-2} \geq k$. Then $k\delta^1 = n^2k\delta^n + (k\delta^1 - n^2k\delta^n)$. The first term is in the convex set $E_f - E_f$ (since $2^{n-1}\delta^n \in E_f - E_f$ and $n^2k \leq 2^{n-1}$) while the second is clearly in M . We have shown that $M + V_f$ is a proper subspace of V'_f ; since $M = M + V_f$ is closed, it is not dense.

In looking for a non-reflexive Banach space which might answer the question posed at the beginning of this section, one naturally turns to either $C(X)$ (X compact Hausdorff) or L_1 of some measure space. It is an interesting exercise to verify the following fact: If E is $C(X)$ or L_1 (over an arbitrary measure space) then the following is true: For $f \in U^*$ let $H_f = \{x \in E: f(x) = \|x\|\}$. Then f is an extreme point of U^* if and only if $E = H_f - H_f$. Furthermore (defining H_x^* in the obvious way), $x \in U$ is an extreme point of U if and only if $E^* = H_x^* - H_x^*$. (Representations of E^* for the above spaces, as well as characterizations of the appropriate extreme points, may be found in [3; 5].)

These results say, in effect, that each extreme point of U or U^* is a "vertex"; in finite dimensional spaces this is true for any space whose unit ball is a (centrally symmetric) polyhedron.

Added in proof. K. de Leeuw has suggested and Y. Katznelson has proved that the answer to the question raised at the beginning of §4 is negative for the Banach space of all complex valued functions which are analytic for $|z| < 1$ and continuous for $|z| \leq 1$, with the supremum norm.

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