ON THE ORDER OF THE DIFFERENCE OF TWO MEROMORPHIC FUNCTIONS

S. M. SHAH

1. **Introduction.** In a recent paper [5] S. K. Singh has proved the following theorem:

THEOREM A. *Let*

$$f(z) = P_1(z) \exp(Q_1(z)) - P_2(z) \exp(Q_2(z)) \not\equiv 0$$

where $Q_1(z)$ and $Q_2(z)$ are polynomials of degree ρ (ρ being an integer) and $P_1(z)$, $P_2(z)$ are canonical products of order $< \rho$; then $\log M(r, f) > Ar^\rho$ where A is some suitable constant; $M(r, f)$ being $\max |f(z)|$ for $|z| = r$.

In this note we prove the following extension of Theorem A.

THEOREM 1. *Let $P_i(z)$, $i=1, 2, 3, 4$ be canonical products or polynomials (not having any zero at origin) or constants (neither zero nor infinity) such that P_1 and P_3 have no common zeros, P_2 and P_4 have no common zeros, and let ρ be a positive integer and*

$$(1.1) \quad \limsup_{r \rightarrow \infty} T(r, P_i)/r^\rho = a_i < \infty.$$

Further let

$$(1.2) \quad Q_1(z) = az^\rho + c_1z^{\rho-1} + \dots, \quad Q_2(z) = bz^\rho + d_1z^{\rho-1} + \dots$$

be two polynomials such that $\max(|a|, |b|) \neq 0$, k_1, k_2 two integers or zeros and

$$(1.3) \quad F(z) = z^{k_1} \{P_1(z)/P_3(z)\} \exp(Q_1(z)) - z^{k_2} \{P_2(z)/P_4(z)\} \exp(Q_2(z)) \not\equiv 0.$$

Write

$$D(F) = \{ \max(|a|, |b|, |a-b|) - \min(|a|, |b|, |a-b|) \} / \pi,$$

$$I(F) = \liminf_{r \rightarrow \infty} T(r, F)/r^\rho, \quad J(F) = \limsup_{r \rightarrow \infty} T(r, F)/r^\rho,$$

$$B_1 = \limsup_{r \rightarrow \infty} T(r, P_1/P_3)/r^\rho, \quad B_2 = \limsup_{r \rightarrow \infty} T(r, P_2/P_4)/r^\rho,$$

$$A_1 = \limsup_{r \rightarrow \infty} T(r, P_1P_4)/r^\rho, \quad A_2 = \limsup_{r \rightarrow \infty} T(r, P_2P_3)/r^\rho,$$

$$A_3 = \limsup_{r \rightarrow \infty} T(r, P_3P_4)/r^\rho.$$

Presented to the Society, February 27, 1960; received by the editors December 29, 1959 and, in revised form, March 14, 1960 and May 6, 1960.

Then

$$(1.4) \quad \iota(F) \geq D(F) - \sum_{i=1}^2 B_i \geq D(F) - \sum_{i=1}^4 a_i,$$

$$(1.5) \quad \iota(F) \geq |a - b|/\pi - \min \left\{ 2 \sum_{i=1}^2 A_i, 3 \sum_{i=1}^2 B_i \right\},$$

$$(1.6) \quad \iota(F) \geq \max\{|a|, |b|\}/\pi - 2 \sum_{i=1}^2 B_i.$$

If further $\sum_{i=1}^2 B_i = 0$ and either $\min(|a|, |b|) = 0$ or $\arg a = \arg b + k\pi$ (k integer or zero) then

$$(1.7) \quad \iota(F) = \Im(F) = \max\{|a|, |b|, |a - b|\}/\pi.$$

We assume here that the right-hand expressions of (1.4)–(1.6) (and of (2.1), (2.3), (2.4) below) are positive, for otherwise these results are trivially true.

J. Clunie [3] proved that if $f(z)$ and $g(z)$ are two integral functions of orders ρ_1 and ρ_2 ($< \rho_1$), then $f'(z)g(z) - f(z)g'(z)$ is of order ρ_1 and raised the question about the order of this function when type of $g(z)$ be less than type of $f(z)$. We prove here

THEOREM 2. Let $f_i(z)$ ($i=1, 2$) be meromorphic functions of order $\leq \rho$ ($\rho > 0$) and write

$$\limsup_{r \rightarrow \infty} T(r, f_i)/r^\rho = \alpha_i,$$

$$\limsup_{r \rightarrow \infty} \{\overline{N}(r, 1/f_i) + \overline{N}(r, f_i)\}/r^\rho = \gamma_i,$$

$$f(z) = f'_1(z)f_2(z) - f_1(z)f'_2(z).$$

Suppose $\alpha_1 > 0$, $\alpha_2 < \infty$. Then (i) $f(z)$ is of order ρ , maximal type if $\alpha_1 = \infty$ and mean type if $\alpha_1 < \infty$, $\alpha_2 = 0$; (ii) $f(z)$ is of order ρ mean type if $\alpha_1 < \infty$ and either (ii.a)

$$\alpha_1 > (1 + 2Ak^{3\rho})\alpha_2; \quad A = \frac{1}{\log k} \log \left(\frac{16ek^2}{k-1} \right) + \frac{2(k+1)}{k-1}, \quad k > 1;$$

or

$$(ii.b) \quad \alpha_1 > \alpha_2 + \gamma_1 + \gamma_2;$$

or

$$(ii.c)^1 \quad 3\alpha_2 < \alpha_1 \{\theta(0, f_1) + \theta(\infty, f_1) - 1\}.$$

¹ For the definitions, see [6, Chapter 2].

2. Proof of Theorem 1.

LEMMA 1. Let $F(z)$ be given by (1.3), and A be any number. Then

$$(2.1) \quad \iota(F) \geq \{ |A| - |A - a| - |A - b| \} / \pi - \sum_{i=1}^2 B_i,$$

$$(2.2) \quad \mathfrak{J}(F) \leq \{ |A| + |A - a| + |A - b| \} / \pi + \sum_{i=1}^2 B_i.$$

PROOF.

$$\begin{aligned} F(z) &= \exp(Az^p) \{ z^{k_1} \exp((a - A)z^p + \dots) P_1/P_3 \\ &\quad - z^{k_2} \exp((b - A)z^p + \dots) P_2/P_4 \} \\ &= \exp(Az^p) \phi(z). \end{aligned}$$

Then

$$\begin{aligned} \{ |A| r^p \} / \pi &\leq T(r, F) + T(r, \phi) + O(1) \\ &\leq T(r, F) + \left\{ \frac{|a - A| + |b - A|}{\pi} + \sum_{i=1}^2 B_i + o(1) \right\} r^p \end{aligned}$$

and (2.1) follows. Also

$$\mathfrak{J}(F) \leq |A| / \pi + \limsup_{r \rightarrow \infty} T(r, \phi) / r^p$$

and (2.2) follows.

LEMMA 2. Let $F(z)$ be given by (1.3). Then

$$(2.3) \quad \iota(F) \geq \{ |a - b| - \min(|a|, |b|) \} / \pi - \sum_{i=1}^2 B_i.$$

PROOF. We may suppose $|b| \leq |a|$. Write

$$X = z^{k_1} \exp\{(a - b)z^p + \dots\} P_1/P_3; \quad Y = z^{k_2} \exp(d_1 z^{p-1} + \dots) P_2/P_4.$$

Then

$$F(z) \exp(-bz^p) = X - Y,$$

$$T(r, X) \leq T(r, F) + \{ |b| r^p \} / \pi + (B_2 + o(1)) r^p.$$

Further

$$\begin{aligned} \exp\{(a - b)z^p\} &= X z^{-k_1} P_3 \exp(-c_1 z^{p-1} + \dots) / P_1, \\ \frac{|a - b| r^p}{\pi} &\leq T(r, F) + \frac{|b| r^p}{\pi} + (B_1 + B_2 + o(1)) r^p \end{aligned}$$

and (2.3) follows.

LEMMA 3. Let $F(z)$ be given by (1.3). Then

$$(2.4) \quad t(F) \geq |a| - |b|/\pi - \sum_{i=1}^2 B_i.$$

We may suppose $|a| > |b|$. Write $F(z) = f_1 - f_2$. Then

$$\begin{aligned} T(r, f_1) &\leq T(r, F) + T(r, f_2) + O(1) \\ &\leq T(r, F) + \left(\frac{|b|}{\pi} + B_2 + o(1) \right) r^\rho. \end{aligned}$$

Further

$$\begin{aligned} \exp(az^\rho + c_1 z^{\rho-1} + \dots) &= z^{-k_1} f_1 P_3 / P_1, \\ \{|a| r^\rho\} / \pi &\leq \{T(r, f_1) + B_1 + o(1)\} r^\rho \end{aligned}$$

and (2.4) follows.

PROOF OF (1.4). We may suppose $|a| \geq |b|$. Then (1.4) follows from (2.1) (with $A=a$), (2.3) and (2.4).

PROOF OF (1.5). Write

$$\begin{aligned} (2.5) \quad X &= z^{k_3} \{P_2 P_3 / P_4 P_1\} \exp\{(b-a)z^\rho + d_1 z^{\rho-1} + \dots\}, \\ Y &= z^{k_1} \exp(c_1 z^{\rho-1} + \dots), \end{aligned}$$

$$(2.6) \quad \phi_2 = (X - Y)P_1/P_3; \quad \phi_1 = \exp(az^\rho).$$

Then $F(z) = -\phi_1 \phi_2$. Let γ_i be a common zero (if any) of P_3 and P_4 with multiplicity $k_{3,i}$ in P_3 and $k_{4,i}$ in P_4 . Write

$$\begin{aligned} N_1(r) &= \sum \min(k_{3,i}, k_{4,i}) \log r / |\gamma_i|, & k_{3,i} \neq k_{4,i}; & |\gamma_i| \leq r, \\ N_2(r) &= \sum k_{3,i} \log r / |\gamma_i|, & k_{3,i} = k_{4,i}; & |\gamma_i| \leq r. \end{aligned}$$

Then it is easily seen that

$$\begin{aligned} (2.7) \quad &N(r, X) + N(r, 1/X) \\ &\leq N(r, 1/P_2 P_3) + N(r, 1/P_4 P_1) - N_1(r) - 2N_2(r) + O(\log r) \\ &\leq \sum_{i=1}^4 N(r, 1/P_i) - N_1(r) - 2N_2(r) + O(\log r), \end{aligned}$$

$$(2.8) \quad N(r, 1/(X - Y)) \leq N(r, 1/\phi_2) + N_2(r) + O(\log r)$$

since only those zeros of P_3 can be zeros of $X - Y$ which are considered in $N_2(r)$. Further

$$(2.9) \quad N(r, \phi_2) \geq \sum_{i=3}^4 N(r, 1/P_i) - N_1(r) - 2N_2(r) + O(\log r).$$

To prove (1.5) we assume that the right-hand expression of (1.5) is positive. Hence either $|a-b| > \pi(A_1+A_2)$ or $|a-b| > \pi(B_1+B_2)$. In both cases we have by an extension of the second theorem of Nevanlinna [4, pp. 75-76]

$$(2.10) \quad T(r, X) \leq N(r, X) + N(r, 1/X) + N(r, 1/(X-Y)) + o(r^\rho).$$

Now

$$\begin{aligned} T(r, F) &\geq T(r, 1/\phi_1\phi_2) + O(1) \\ &\geq N(r, 1/\phi_2) + O(1) \\ (2.11) \quad &\geq N(r, 1/(X-Y)) \\ &\geq T(r, X) - N(r, X) - N(r, 1/X) - N_2(r) + o(r^\rho) \\ &\geq \{ |a-b| r^\rho \} / \pi - 2T(r, P_2P_3) - 2T(r, P_4P_1) + o(r^\rho). \end{aligned}$$

Also from (2.11)

$$\begin{aligned} T(r, F) &\geq \{ |a-b| r^\rho \} / \pi - T(r, P_1/P_3) - T(r, P_2/P_4) \\ &\quad - \sum_{i=1}^4 N(r, 1/P_i) + o(r^\rho) \\ &\geq \{ |a-b| r^\rho \} / \pi - 3T(r, P_1/P_3) - 3T(r, P_2/P_4) + o(r^\rho) \end{aligned}$$

and (1.5) follows.

PROOF OF (1.6). We may suppose $|a| \geq |b|$.

Case (i): $|a-b| \leq \pi(B_1+B_2)$.

Then from Lemma 1 we have with $A=a$,

$$t(F) \geq \frac{|a| - |a-b|}{\pi} - \sum_{i=1}^2 B_i \geq \frac{|a|}{\pi} - 2 \sum_{i=1}^2 B_i.$$

Case (ii): $|a-b| > \pi(B_1+B_2)$. We have in the notation of the proof of (1.5)

$$\begin{aligned} T(r, F) &\geq m(r, 1/\phi_1\phi_2) + N(r, 1/\phi_2) + O(1) \\ &\geq \{ |a| r^\rho \} / \pi - T(r, \phi_2) + N(r, \phi_2) + N(r, 1/\phi_2) + O(1) \\ &\geq \{ |a| r^\rho \} / \pi - T(r, \phi_2) + N(r, 1/(X-Y)) - 3N_2(r) \\ &\quad + \sum_{i=3}^4 N(r, 1/P_i) - N_1(r) + O(\log r) \\ (2.12) \quad &\geq \{ |a| r^\rho \} / \pi - T(r, \phi_2) + T(r, X) - N(r, X) - N(r, 1/X) \\ &\quad - 3N_2(r) - N_1(r) + \sum_{i=3}^4 N(r, 1/P_i) + o(r^\rho) \\ &\geq \{ |a| r^\rho \} / \pi - T(r, P_1/P_3) - \sum_{i=1}^2 N(r, 1/P_i) - N_2(r) + o(r^\rho). \end{aligned}$$

Since $N_2(r) \leq N(r, 1/P_4)$, we have

$$t(F) \geq |a|/\pi - 2(B_1 + B_2).$$

PROOF OF (1.7). From (2.2) we have

$$(2.13) \quad \begin{aligned} \mathfrak{J}(F) &\leq \min\{|a| + |a - b|, |b| + |a - b|, |a| + |b|\}/\pi \\ &\quad + \sum_{i=1}^2 B_i. \end{aligned}$$

From the hypothesis on a and b we have either

$$(i) \quad |a - b| = |a| + |b| \quad \text{or} \quad (ii) \quad |a - b| = ||a| - |b||.$$

In case (i) we have from (2.13), (1.5), (1.6), since $\sum_{i=1}^2 B_i = 0$,

$$t(F) \geq \{|a| + |b|\}/\pi \geq \mathfrak{J}(F)$$

and so (1.7) follows. In case (ii) we have from (2.13), (1.5), (1.6)

$$t(F) = \mathfrak{J}(F) = \max\{|a|, |b|\}/\pi = \max\{|a|, |b|, |a - b|\}/\pi.$$

3. Remarks on Theorem 1. If we write

$$F(z) = (z^{k_1} \exp(Q_1)P_1P_4 - z^{k_2} \exp(Q_2)P_2P_3)/P_3P_4$$

we get

$$(3.1) \quad t(F) \geq \frac{1}{\pi} \max(|a|, |b|) - 2A_1 - 2A_2 - A_3,$$

$$(3.2) \quad \begin{aligned} \mathfrak{J}(F) &\leq \min\{|a| + |a - b|, |b| + |a - b|, |a| + |b|\}/\pi \\ &\quad + \sum_{i=1}^3 A_i. \end{aligned}$$

If $|a - b| > \pi \min(\sum_{i=1}^2 B_i, \sum_{i=1}^2 A_i)$, then we have from (2.12)

$$(3.3) \quad t(F) \geq \frac{|a|}{\pi} - \min\{2a_1 + a_2 + 2a_3, 2a_1 + a_2 + a_3 + a_4\}$$

and similarly

$$(3.4) \quad t(F) \geq \frac{|b|}{\pi} - \min\{a_1 + 2a_2 + 2a_4, a_1 + 2a_2 + a_3 + a_4\}.$$

The inequalities (3.1), (3.2) and (1.5) (with $\sum A_i$) give an interval $I(A)$ in which $t(F)$ and $\mathfrak{J}(F)$ lie; and (1.4), (1.5) (with $\sum B_i$), (1.6) and (2.13) give another such interval $I(B)$. It is possible to choose a function F for which $I(A) \subset I(B)$ and another function F for which $I(B) \subset I(A)$.

EXAMPLE 1. Write

$$\alpha_n = (-1)^{n+1}(n+1) \log(n+1), \quad \sum_{n=1}^{\infty} 1/\alpha_n = \alpha,$$

$$P_1(z) = P_2(z) = \prod_{n=1}^{\infty} E(z/\alpha_n, 1), \quad P_3(z) = P_4(z) = \prod_{n=1}^{\infty} E(z/(-\alpha_n), 1).$$

Let

$$a > 0, k > 0, ak > 8\alpha, \quad F(z) = (e^{az} - e^{-k\alpha z})P_1/P_3.$$

By a known theorem [1, p. 27] P_1P_3 is a function of zero type order 1. We have

$$A_1 = A_2 = 0, \quad A_3 = 2\alpha/\pi = B_1 = B_2;$$

$$I(A): \{a(1+k)/\pi, (a(1+k) + 2\alpha)/\pi\},$$

$$I(B): \{(a(1+k) - 12\alpha)/\pi, (a(1+k) + 4\alpha)/\pi\}.$$

EXAMPLE 2. Let

$$(\beta_k)_1^{\infty}$$

be such that $|\beta_1| < |\alpha_1| < |\beta_2| < |\alpha_2| < \dots$, $\alpha_n/\beta_n > 0$, $\sum_{n=1}^{\infty} 1/\beta_n = \alpha$, where α_n, α have been defined in Example 1. Let

$$P_3^*(z) = \prod_{n=1}^{\infty} E(z/\beta_n, 1), \quad F(z) = (e^{az} - e^{-k\alpha z})P_1/P_3^*$$

where $a > 0, k > 0, ak > 8\alpha$ and P_1 is as in Example 1. Then

$$B_1 = B_2 = 0, \quad A_1 = A_2 = A_3 = 2\alpha/\pi.$$

Hence $t(F) = \mathfrak{I}(F) = a(1+k)/\pi$ as given by (1.7); whereas $I(A)$ is

$$\{(a(1+k) - 8\alpha)/\pi, (a(1+k) + 6\alpha)/\pi\}.$$

4. PROOF OF THEOREM 2. (i) We have

$$f = f_2^2 \frac{d}{dz} (f_1/f_2) = f_2^2 (f_1/f_2)'.$$

Further [2] if $k > 1, r > r_0$, then²

$$(4.1) \quad T(r, f_1/f_2) < AT(k^3 r, (f_1/f_2)') + k_1 \log r$$

where $A = A(k)$ is defined in the statement of Theorem 2, and k_1 is a constant. Hence

² The relation (4.1) follows from the inequalities (26), (5), (6) of [2].

$$\begin{aligned}
 T(r, f) &> T(r, (f_1/f_2)') - 2T(r, f_2) + O(1) \\
 (4.2) \quad &> \frac{1}{A} T(r/k^3, f_1/f_2) - 2T(r, f_2) + O(\log r) \\
 &> \frac{1}{A} \{T(r/k^3, f_1) - T(r/k^3, f_2)\} - 2T(r, f_2) + O(\log r).
 \end{aligned}$$

Also for $r > r_0(\epsilon)$

$$\begin{aligned}
 T(r, f) &< 2T(r, f_2) + T(r, (f_1/f_2)') + O(1) \\
 (4.3) \quad &< 2T(r, f_2) + (2 + \epsilon)T(r, f_1/f_2) + O(1) \\
 &< (2 + \epsilon)T(r, f_1) + (4 + \epsilon)T(r, f_2) + O(1).
 \end{aligned}$$

Hence (i) follows. Further from (4.2) we have, if $\alpha_1 < \infty$, for a sequence of values of $r \uparrow \infty$,

$$T(r, f) > \frac{1}{A} \{(\alpha_1 - \alpha_2)(r/k^3)^\rho\} - 2\alpha_2 r^\rho - o(r^\rho);$$

and so

$$(4.4) \quad \mathfrak{J}(F) \geq \{\alpha_1 - \alpha_2(1 + 2Ak^{3\rho})\}/Ak^{3\rho} > 0.$$

From (4.3) we get

$$\mathfrak{J}(f) \leq 2\alpha_1 + 4\alpha_2$$

and (ii.a) is proved.

(ii.b–ii.c) We have

$$f = f_1 f_2 (f_1'/f_1 - f_2'/f_2) \equiv f_1 f_2 \phi.$$

Then

$$\begin{aligned}
 T(r, f_1) - T(r, f_2) &\leq T(r, f_1 f_2) + O(1) \\
 &\leq T(r, f) + T(r, \phi) + O(1).
 \end{aligned}$$

Further

$$\begin{aligned}
 T(r, \phi) &\leq T(r, f_1'/f_1) + T(r, f_2'/f_2) + O(1) \\
 &= O(\log r) + N(r, f_1'/f_1) + N(r, f_2'/f_2) \\
 &\leq O(\log r) + \sum_{i=1}^2 \{\overline{N}(r, 1/f_i) + \overline{N}(r, f_i)\}.
 \end{aligned}$$

Hence

$$(4.5) \quad \begin{aligned} T(r, f) &\geq T(r, f_1) - T(r, f_2) \\ &\quad - \sum_{i=1}^2 \{ \overline{N}(r, 1/f_i) + \overline{N}(r, f_i) \} + O(\log r) \end{aligned}$$

and so

$$\mathfrak{I}(f) \geq \alpha_1 - \alpha_2 - \gamma_1 - \gamma_2 > 0.$$

Further from (4.5) we have for $r > r_0(\epsilon)$,

$$\begin{aligned} T(r, f) &\geq \{ \theta(0, f_1) + \theta(\infty, f_1) - 1 \} T(r, f_1) - 3\alpha_2 r^\rho - o(r^\rho), \\ \mathfrak{I}(f) &\geq \{ \theta(0, f_1) + \theta(\infty, f_1) - 1 \} \alpha_1 - 3\alpha_2 > 0. \end{aligned}$$

Added in proof. (i) The author wishes to thank the referee for his comments. (ii) Write $|a| = \alpha$, $|b| = \beta$, $|a - b| = \gamma$. If we suppose that $\alpha^2 \leq \beta^2 + \gamma^2 + \beta\gamma$, $\beta^2 \leq \gamma^2 + \alpha^2 + \gamma\alpha$, $\gamma^2 \leq \beta^2 + \alpha^2 + \beta\alpha$, then we can get, from (2.2), a sharper inequality for $\mathfrak{I}(F)$, than (2.13) or (3.2). In fact we can show, in this case, that

$$\mathfrak{I}(F) \leq \frac{1}{\pi} \left\{ \frac{\sum \alpha^2 + (6 \sum \alpha^2 \beta^2 - 3 \sum \alpha^4)^{1/2}}{2} \right\}^{1/2} + \min \left(\sum_1^2 B_i, \sum_1^3 A_i \right)$$

where $\sum \alpha^r = \alpha^r + \beta^r + \gamma^r$.

REFERENCES

1. R. P. Boas, Jr., *Entire functions*, New York, Academic Press, 1954.
2. Chi-Tai Chuang, *Sur la comparaison de la croissance d'une fonction Meromorphe et de celle de sa dérivée*, Bull. Sci. Math. vol. 75 (1951) pp. 171-190.
3. J. Clunie, *The derivative of a meromorphic function*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 227-229.
4. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions meromorphes*, Paris, Gauthier-Villars, 1929.
5. S. K. Singh, *A note on entire and meromorphic functions*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 6-10.
6. H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Berlin, Springer, 1955.

NORTHWESTERN UNIVERSITY, AND
MUSLIM UNIVERSITY, ALIGARH, INDIA