ON GRADIENT MAPPINGS IN BANACH SPACES

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1. Introduction. Let E be a real Banach space, V a convex open subset of E . A real valued functional $I(x)$ defined on V is said to have the Fréchet derivative $D(x, h)$ if, for arbitrary fixed $x \in V$, $D(x, h)$ is a bounded linear functional on E in the variable h and

$$(1) \quad I(x + h) - I(x) = D(x, h) + \omega(x, h)$$

where

$$\omega(x, h) = o(\|h\|).$$

Thus $x \rightarrow D(x, \cdot)$ is a mapping of V into the space E^* conjugate to E , and is called a gradient mapping. It is said to be compact, if the image of each bounded set of V has a compact closure in E^* .

Recently E. H. Rothe [1] gave a (necessary and) sufficient condition for the compactness of gradient mappings under the condition that E has property (P), and showed that every reflexive Banach space with a basis has it. Here E is said to have property (P), if there exists a sequence $\{\psi_i\}$ of linearly independent elements of E^* and a positive number M with the following properties: the closed linear span of $\{\psi_i\}$ coincides with E^* and for each positive integer n there exists a linear projection of norm at most M on the intersection $\bigcap_{i=1}^n N_i$ where $N_i = \{x \in E \mid \psi_i(x) = 0\}$. In this note a theorem with the conclusion that the gradient mapping is compact is proved under conditions differing somewhat from those of Rothe, and without using anything like the property (P).

THEOREM. *Let $I(x)$ have the following two properties: (a) for each $\alpha > 0$ there exists a positive number β such that*

$$|I(x) - I(y)| \leq \beta \|x - y\|, \quad x, y \in V, \|x\|, \|y\| \leq \alpha;$$

(b) to each $\epsilon > 0$ corresponds a finite number of elements $\phi_1, \phi_2, \dots, \phi_n$ of E^ such that*

$$|I(x + h) - I(x)| \leq \epsilon \|h\|, \quad x, x + h \in V,$$

for all h for which $\|h\| \leq \epsilon$ and $\phi_i(h) = 0$ ($i = 1, 2, \dots, n$). Then the gradient mapping $D(x, \cdot)$ is compact.

2. Proof of theorem. Our proof is based on the following decomposition lemma instead of on the use of linear projections.

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LEMMA. Let N be a closed linear subspace of E with a finite co-dimension. Then there exists a compact set A with the properties that $\|x\| \leq 1$ ($x \in A$), and each $h \in S$ (the open unit ball) admits a decomposition (not unique) $h = a + b$ with $a \in A$, $b \in N$.

PROOF. We consider as usual the quotient space E/N with the norm defined by $\|P(x)\| = \inf_{y \in N} \|x - y\|$ where $x \rightarrow P(x)$ is the canonical mapping of E onto E/N . Since E/N is finite dimensional, the image $P(S)$ of S has a compact closure in it. We shall construct by induction a sequence $\{a_{ij}\}$, $i = 0, 1, 2, \dots$, $j = 1, 2, \dots, k_i$, of elements of S such that

$$(2) \min_k \|a_{ik} - a_{i+1,j}\| < 1/2^i, \quad i = 0, 1, 2, \dots, j = 1, 2, \dots, k_{i+1},$$

and for each $x \in S$ and each i there exists $y \in E$ (depending on x and i) such that

$$(3) \quad P(x) = P(y) \quad \text{and} \quad \min_j \|a_{ij} - y\| < 1/2^i.$$

Put $k_0 = 1$ and $a_{01} = O$ (the origin). Suppose that

$$\{a_{ij}\}, \quad i = 0, 1, 2, \dots, n, \quad j = 1, 2, \dots, k_i,$$

have been constructed. Denote, for convenience, by $S_n(x)$ the open ball with center x and radius $1/2^n$. Since $P(S_n(a_{nj}) \cap S)$ is totally bounded in E/N and $P(S) \subset \bigcup_j P(S_n(a_{nj}))$ because of (3), there exists a finite number of elements of S , denoted by $a_{n+1,j}$, $j = 1, 2, \dots, k_{n+1}$, such that $a_{n+1,j} \in \bigcup_k S_n(a_{nk})$, $j = 1, 2, \dots, k_{n+1}$, and

$$\min_j \|P(x) - P(a_{n+1,j})\| < 1/2^{n+1} \quad \text{for all } x \in S.$$

Then on account of the definition of norm in E/N , for each $x \in S$ there can be chosen $y \in E$ such that $P(x) = P(y)$ and $\min_j \|a_{n+1,j} - y\| < 1/2^{n+1}$. Thus the induction is completed. Notice that the set $\{P(a_{ij})\}$ is dense in $P(S)$ by construction. We claim that the closure A of the set $\{a_{ij}\}$ is compact. In fact, from (2) it follows that for each n , m with $m \geq n$

$$\min_k \|a_{mj} - a_{nk}\| < \sum_{i=n}^m 1/2^i < 1/2^{n-1}.$$

Since $\{a_{ij} | i \leq n\}$ is a finite set, this means that the set $\{a_{ij}\}$ is totally bounded; consequently its closure A is compact. Since $P(A)$ is compact in E/N and contains a dense subset of $P(S)$, it follows that $P(A) \supset P(S)$, that is, for each $h \in S$ there exists $a \in A$ with $P(a) = P(h)$, i.e., $h - a \in N$.

Now we turn to the proof of the theorem. If $D(x, \cdot)$ is not compact, there exist a positive number ϵ and a sequence $\{x_i\}$ of elements of V such that $\|x_i\| \leq 1/\epsilon$, $i = 1, 2, \dots$, and $\|D(x_i, \cdot) - D(x_j, \cdot)\| > \epsilon$ for $i \neq j$. Let $\phi_1, \phi_2, \dots, \phi_n$ be chosen as in (b) with $\epsilon/8$ instead of ϵ . Since $N = \{x \in E \mid \phi_i(x) = 0, i = 1, 2, \dots, n\}$ has a finite co-dimension, by the lemma there exists a compact set A of S^- (the closure of S) such that each $h \in S$ admits a decomposition $h = a + b$ with $a \in A$, $b \in N$. It follows that $\|b\| \leq \|a\| + \|h\| \leq 2$. Since property (a) combined with definition (1) implies $\sup_i \|D(x_i, \cdot)\| < \infty$, the family $\{D(x_i, h)\}$, considered as continuous functions on the compact set A in the variable h , is equi-continuous; consequently Ascoli-Arzelà's theorem tells us that there exists a subsequence $\{x'_i\}$ for which

$$\sup_{a \in A} |D(x'_i, a) - D(x'_j, a)| < \epsilon/2, \quad i, j = 1, 2, \dots$$

On the other hand, on account of the definition of norm of linear functionals,

$$\begin{aligned} \|D(x'_i, \cdot) - D(x'_j, \cdot)\| &= \sup_{h \in S} |D(x'_i, h) - D(x'_j, h)| \\ &\leq \sup_{a \in A} |D(x'_i, a) - D(x'_j, a)| + 2 \cdot \sup_{k, b \in N, \|b\| \leq 2} |D(x'_k, b)|. \end{aligned}$$

Since $b \in N$ and $\|b\| \leq 2$, property (b) (with $\epsilon/8$) combined with (1) implies $|D(x'_k, b)| \leq \epsilon/4$, consequently

$$\|D(x'_i, \cdot) - D(x'_j, \cdot)\| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This contradiction establishes the theorem.

REFERENCE

1. E. H. Rothe, *A note on gradient mappings*, Proc. Amer. Math. Soc. vol. 10 (1959) pp. 931-935.

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