ON BINARY SEQUENCES¹

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Let $x_i = \pm 1$, for $i = 1, \dots, n$ and let $c_k = \sum_{i=1}^{n-k} x_i x_{i+k}$. We shall be concerned with sequences (x_i) for which $|c_k| \leq 1$, $0 < k < n$. The problem was first considered in [1]. The following sequences with this property are known:

$n = 2$	++
3	++-
4	+++ - ; +-+-
5	++++-
7	++++--+-
11	++++-----+-
13	+++++---++--

where it has been assumed $x_1 = x_2 = +1$. Each one of these gives rise to three others under the transformations

$$(1) \quad \begin{aligned} x'_i &= (-1)^i x_i, \\ x'_i &= (-1)^{i+1} x_i, \\ x'_i &= -x_i. \end{aligned}$$

The above sequences have the following interpretation [2]: the elements which are -1 are those with subscript given mod n by

$$\begin{aligned} n = 5: \quad m^4 + 3, \quad m \not\equiv 0 \pmod{5}, \\ n = 7: \quad m^2 + 3, \quad m \not\equiv 0 \pmod{7}, \\ n = 11: \quad -m^2 - 2, \quad (\text{mod } 11), \\ n = 13: \quad 11m^4 - 1, \quad (\text{mod } 13). \end{aligned}$$

It will be shown here that for odd n , these are all the sequences with the property $|c_k| \leq 1$. For the sake of completeness, the preliminary details have been included.

Let (x_i) be any sequence with $x_i = \pm 1$, and let

Received by the editors May 26, 1960.

¹ The research reported in this paper has been partially sponsored by the Electronics Research Directorate of the AFCRC, Air Research and Development Command, under Contract AF-19(604)-5473.

$$c_k = \sum_1^{n-k} x_i x_{i+k}.$$

Since the terms $x_i x_{i+k}$ are each ± 1 , it follows that:

$$(2) \quad \prod_1^{n-k} (x_i x_{i+k}) = (-1)^{(n-k-c_k)/2}, \quad 0 \leq k < n.$$

Multiplying two successive equations of this form gives

$$x_{n-k} x_{k+1} = (-1)^{n-k-(1+c_k+c_{k+1})/2}, \quad 1 \leq k \leq n-1.$$

Now consider

$$c_k + c_{n-k} = \sum_1^n x_i x_{i+k}$$

(the second index being taken mod n). Then (2) yields

$$\prod_1^n (x_i x_{i+k}) = 1 = (-1)^{(n-c_k-c_{n-k})/2}.$$

Thus $c_k + c_{n-k} \equiv n \pmod{4}$.

We now assume $|c_k| \leq 1$, $0 < k < n$; thus $c_k = 0$ for $n-k$ even, $c_k = \pm 1$ for $n-k$ odd.² Now assume also that n is odd. In this case, one of c_k and c_{n-k} is odd and the other case even, hence 0. Thus

$$c_{2i+1} = 0, \\ c_{2j} = (-1)^{(n-1)/2},$$

the latter since $c_k + c_{n-k} \equiv n \pmod{4}$. The formula for $x_{n-k} x_{k+1}$ becomes

$$(3) \quad x_{n-k} x_{k+1} = (-1)^{(n-1)/2+k}.$$

A sequence which satisfies (3) has the property that $c_{2j+1} = 0$, $0 < 2j+1 < n$.

$$c_{2j} = \sum_1^{n-2j} x_i x_{i+2j}, \quad 0 < 2j < n,$$

or

$$(-1)^{(n-1)/2} = \sum_1^{n-2j} x_i x_{n-2j+1-i} (-1)^{(n-1)/2+i+1}.$$

Let $n-2j = 2k+1$, $k \geq 1$.

² It follows from this that $n \equiv 2 \pmod{4}$ implies $n=2$. For if $n > 2$, $c_2 = c_{n-2} = 0$, $c_2 + c_{n-2} = 0 \not\equiv 2 \pmod{4}$. Thus n even, $n > 2$ implies $n \equiv 0 \pmod{4}$, $c_k + c_{n-k} = 0$.

$$\begin{aligned}
 1 &= \sum_{i=1}^{2k+1} x_i x_{2k+2-i} (-1)^{i+1} \\
 &= 2 \sum_{i=1}^k x_i x_{2k+2-i} (-1)^{i+1} + x_{k+1}^2 (-1)^{k+2}, \quad n > 2k+1 \geq 3.
 \end{aligned}$$

Hence

$$(k) \quad \frac{1 + (-1)^{k+1}}{2} = \sum_{i=1}^k x_i x_{2k+2-i} (-1)^{i+1} \quad 1 \leq k < \frac{n-1}{2}.$$

We shall write $P(k)$ for $(1 + (-1)^{k+1})/2$.

THEOREM 1. *Let (x_i) satisfy equations (k) for $1 \leq k \leq t$, $x_i = \pm 1$. Let $x_i = 1$ for $1 \leq i \leq p$, $x_{p+1} = -1$; assume $p > 1$. Then*

- (i) $x_i x_{i+1} = x_2 x_{2i+1}$, $1 \leq i \leq t$,
- (ii) $p \leq 2t+1$ implies p is odd,
- (iii) $pj+r \leq 2t+1$, $1 \leq r \leq p$ implies $x_{p(j-1)+r} = x_{p(j-1)+1}$, and
- (iv) $z_j = x_{p(j-1)+1}$ satisfy equations (k) for $k \leq t/p$.

PROOF. A count of $+1$'s and -1 's in equation (k) shows as before that

$$\prod_{i=1}^k (x_i x_{2k+2-i}) = 1$$

or

$$\prod_{i=1}^{2k+1} x_i = x_{k+1}.$$

Multiplying two successive equations of this form proves the first statement. The second statement follows from the first: for if $p > 2$ were even, say $p = 2s$, we would have

$$x_s x_{s+1} = x_{2s} x_{2s+1}$$

or, by assumption, $1 = -1$.

We prove (iii) and (iv) by induction on t . From $t < p$, there is nothing to prove. For $p = t$, we have:

$$(p) \quad 1 = x_1 x_{2p+1} - x_2 x_{2p} + \cdots + x_p x_{p+2}.$$

Now $x_i = 1$ for $i \leq p$, and by the first part of the theorem we have $x_{2k} = x_{2k+1}$ for $p < 2k < 2p$. Thus the above equation reduces to

$$1 = x_{2p+1} - x_{2p} + x_{p+2}.$$

But also

$$-1 = x_p x_{p+1} = x_{2p} x_{2p+1},$$

so we have

$$1 = 2x_{2p+1} + x_{p+2}.$$

Thus

$$x_{p+2} = -1,$$

$$x_{2p+1} = 1.$$

Similarly, for $2t+1 \leq 3p$, we let $t=p+s$, $s>0$; we have

$$P(p+s) = \sum_1^{2s+1} + \sum_{2s+2}^p + \sum_{p+1}^{p+s} = x_{2p+1} - x_{p+1} x_{p+2s+1} + P(s-1),$$

using the fact that $x_{2i} = x_{2i+1}$ for $i \not\equiv 0 \pmod{p}$ by the induction assumption and (i). Since p is odd, $P(p+s) = P(s-1)$, and

$$x_{2p+1} + x_{p+2s+1} = 0$$

or

$$x_{p+2s+1} = -1.$$

For $2t+1 > 3p$, we let $2t+1 = hp+m$ with $1 \leq m \leq p$, and consider two cases:

(A) h odd, $h=2H+1$. We have

$$\sum_1^t = \sum_0^{H-1} \left[\sum_{jp+1}^{jp+m} + \sum_{jp+m+1}^{(j+1)p} \right] + \sum_{Hp+1}^{Hp+m} + \sum_{Hp+m+1}^t.$$

As before, we reduce the first three sums by using the fact that $x_{2i} = x_{2i+1}$ for $i \not\equiv 0 \pmod{p}$; the subsequent sums are of the form $(-1)^i$ by the induction assumption. Thus we get

$$\begin{aligned} P(t) &= x_p x_{(h-1)p+m+1} - x_{p+1} x_{(h-1)p+m} + x_{p+m} x_{(h-1)p+1} \\ &\quad + \sum_2^H z_i z_{h+1-i} (-1)^{i+1} + (-1)^H P\left(\frac{p+m-1}{2}\right). \end{aligned}$$

We have $t = Hp + (p+m-1)/2$. Therefore

$$P(H) - \sum_1^H z_i z_{h+1-i} (-1)^{i+1} = x_{(h-1)p+m+1} + x_{(h-1)p+m} - 2x_{(h-1)p+1}.$$

By the induction assumption, the left-hand side vanishes, and thus

$$x_{(h-1)p+m} = x_{(h-1)p+1}.$$

(B) If h is even, $h=2H$, we get, proceeding as before,

$$P(H) - \sum_1^H z_i z_{h+1-i} (-1)^{i+1} = -z_h - x_{hp} + 2x_{(h-1)p+m}$$

for $m < p$. Subtracting successive equations of this type (i.e., $m=1, 3, \dots$) gives $x_{(h-1)p+1} = x_{(h-1)p+m}$ (for m odd, and the equation for m even follows from (ii)). For $m=p$, i.e., $2t+1=(2H+1)p$ we get

$$P(H) - \sum_1^H z_i z_{h+1-i} (-1)^{i+1} = z_h + x_{hp}.$$

Comparing this with the equation for $m=1$, we deduce

$$P(H) - \sum_1^H z_i z_{h+1-i} (-1)^{i+1} = 0$$

and hence

$$x_{hp} = x_{(h-1)p+1}.$$

We have shown that a sequence of odd length with $|c_k| \leq 1$ for $k > 0$ has some periodic behavior. We now conclude that it must therefore be short.

THEOREM 2. Let $x_i = \pm 1$, $1 \leq i \leq n$, n odd. Assume $|c_k| \leq 1$ for $k > 0$. Then $n \leq 13$.

PROOF. By using the transformations (1), we may assume that $x_1 = x_2 = 1$. We have equations (k) for $1 \leq k < (n-1)/2$, and clearly p (as defined in Theorem 1) is less than $n-1$ for $n > 3$. First, we must have $n < 4p$: for if $n > 4p$, we would conclude by Theorem 1 that $x_i = +1$, $1 \leq i \leq p$, $2p+1 \leq i \leq 3p$ and $x_i = -1$, $p+1 \leq i \leq 2p$. But if $x_i = x_{i+1}$, $x_{n+1-i} = -x_{n-i}$ by (3), and runs of equal x_i are opposite runs of alternating ones. $n > 4p$ would imply 3 initial constant blocks, hence 3 final alternating blocks, and therefore $n \geq 6p-1$. Similarly, for $k \geq 2$ Theorem 1 shows that $n > kp$ implies $n \geq 2(k-1)p-3$. Thus $n < 4p$. We also know $n \neq 3p$, for $n = 3p$ would imply

$$x_p x_{p+1} = x_{2p} x_{2p+1} = -x_{p+1} x_p.$$

Let $a = (-1)^{(n-1)/2}$, $n-3 = 2m$. Then equation (m) is

$$\frac{1+a}{2} = \sum_1^m x_i x_{n-1-i} (-1)^{i+1}.$$

The first $p-2$ terms in the sum are equal to a . Let N be the number of terms $x_i x_{n-1-i} (-1)^{i+1} = -a$ with $p-2 < i \leq m$. Then

$$\frac{1+a}{2} = (p-2)a + (m-N-(p-2))a - Na,$$

$$\frac{a+1}{2} + N - (p-2) = m - N - (p-2) \geq 0,$$

$$p-2 - \frac{a+1}{2} \leq N \leq \frac{n-3}{2} - (p-2),$$

or

$$4p \leq n + 6 + a.$$

Hence $n < 3p$ implies $p < 6 + a$ or $p \leq 5$. By Theorem 1, we can construct the sequences of lengths 5, 7, 13. Finally, for $3p < n < 4p$, let $n = 2m + 1$. $x_m = -x_{m+2}$, and we have $x_i = -1$ for $p+1 \leq i \leq 2p$. By Theorem 1, we must have $n-2-p \leq m+1 = (n+1)/2$, or $n \leq 2p+5$. Now $n > 3p$ implies $p < 5$, and hence $p=3$ and we reconstruct the sequence of length 11.

REFERENCES

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