A GENERALIZATION OF THE CONCEPT OF ABSOLUTE RETRACT

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1. Introduction. In this paper the notion of absolute retract [4; 5] is generalized for compact Hausdorff spaces to a new concept which we call absolute homotopy retract (AHR). This concept AHR is a topological invariant and is characterized in two ways by Theorems (3.4) and (3.6). An example (3.5) is given which shows that an AHR need not be an absolute retract (AR), nor even an absolute neighborhood retract (ANR) [5]. A product theorem (3.7) is given for AHR. Finally, a characterization theorem (4.3) is given for absolute retracts (AR). The word *mapping* will always mean *continuous mapping*.

2. Homotopy retraction and extension. DEFINITION (2.1). Given the sets A and B such that $B \subset A$, we say that the mapping f is a *homotopy retraction* (h -retraction) of A into B provided $f: A \rightarrow B$ (f is mapping of A into B) and $f|_B \simeq i_B$ (restriction of f to B is homotopic to the identity mapping i_B of B onto B). The set B is called a *homotopy retract* (h -retract) of A .

In his definition of h -retraction, Bartholomay [1] required that f be a mapping of A onto B . However, it is necessary that f be defined as a mapping of A into B as stated in definition (2.1) in order to obtain the theory as developed in this paper. Clearly if B is a retract [2] of A , then B is a homotopy retract of A . Homotopy extension of a mapping was defined by Bartholomay [1] as follows.

DEFINITION (2.2). Given the sets A and B such that $B \subset A$, and a mapping $f: B \rightarrow Y$ (into), we say that the mapping g is a *homotopy extension* (h -extension) of the mapping f over A relative to Y provided $g: A \rightarrow Y$ (into) and $g|_B \simeq f$ (restriction of g to B is homotopic to f).

If $f: A \rightarrow Y$ and $g: A \rightarrow Y$ and $f \simeq g$, then for clarity we will sometimes write $f \simeq g: A \rightarrow Y$. The following well-known result [6, p. 3] will be needed.

LEMMA (2.3). If $f_1 \simeq f_2: X \rightarrow Y$, $g: Y \rightarrow W$ and $h: Z \rightarrow X$, then $gf_1 \simeq gf_2: X \rightarrow W$, and $f_1h \simeq f_2h: Z \rightarrow Y$.

LEMMA (2.4). If $f: P \rightarrow A$ and $P \subset P_1$, and f admits an h -extension g

Presented to the Society, January 25, 1961 under the title *Homotopy retraction properties for compact Hausdorff spaces*; received by the editors March 30, 1960 and April 8, 1960 and, in revised form, June 24, 1960.

over P_1 relative to A , then f admits an h -extension over P_1 relative to any h -retract B of A provided $f(P) \subset B$.

PROOF. By (2.1) there is mapping $r: A \rightarrow B$ such that $r|_B \simeq i_B: B \rightarrow B$. Since $f(P) \subset B$, we can regard f as mapping $f: P \rightarrow B$. By (2.3) therefore $i_B f \simeq (r|_B) f: P \rightarrow B$ or $f \simeq r f: P \rightarrow B$. Now (2.2) implies $(g|_P) \simeq f: P \rightarrow A$. Using (2.3) we have $r(g|_P) \simeq r f: P \rightarrow B$. By transitivity, $(rg)|_P \simeq f: P \rightarrow B$. Since $rg: P_1 \rightarrow B$, we have rg is h -extension of f over P_1 relative to B .

3. Absolute homotopy retract. DEFINITION (3.1). A compact Hausdorff space X is said to be an *absolute homotopy retract* (AHR) provided every topological image X_1 of X , which is a subset of a compact Hausdorff space Y , is an h -retract of Y .

From (3.1) two immediate results follow.

THEOREM (3.2). *The property of being an AHR is a topological invariant.*

THEOREM (3.3). *Any compact absolute retract [4; 5] is an AHR.*

THEOREM (3.4). *A necessary and sufficient condition for a space A to be an AHR is that A be homeomorphic to a closed h -retract of some compact parallelotope Q [3].*

PROOF. NECESSITY. Since A is a compact Hausdorff space, by the Tychonoff imbedding theorem [3, p. 29], there exists a homeomorphism $h: A \rightarrow A_1$ which maps A onto a subset A_1 of a compact parallelotope Q . Since Q is a compact Hausdorff space [3, p. 29], by (3.1) we have A_1 is an h -retract of Q . The continuity of h and compactness of A imply A_1 is compact and therefore closed in Q .

SUFFICIENCY. Let h be a homeomorphism such that $h(A) = A_1$ where A_1 is closed h -retract of some compact parallelotope Q . Consider any other topological image of A , say $k(A) = A_2$ where A_2 is subset of compact Hausdorff space Y . A_1 closed in Q implies A_1 compact and hence A is compact Hausdorff space and A_2 is compact and hence closed in Y . Since $hk^{-1}: A_2 \rightarrow Q$ where Q is compact parallelotope, we apply Tietze's extension theorem [4, p. 1052, proof of Theorem (3.2)] and obtain extension f of hk^{-1} over Y relative to Q . Now A_1 is h -retract of Q implies there exists a mapping $g: Q \rightarrow A_1$ such that $g|_{A_1} \simeq i: A_1 \rightarrow A_1$. Therefore we have mapping $H: A_1 \times (0, 1) \rightarrow A_1$ such that $H(a_1, 0) = g(a_1)$ and $H(a_1, 1) = a_1$. Define mapping $r: Y \rightarrow A_2$ by $r(y) = kh^{-1}gf(y)$ for $y \in Y$. Define mapping $K: A_2 \times (0, 1) \rightarrow A_2$ by $K(a_2, t) = kh^{-1}H(f(a_2), t)$ for $a_2 \in A_2$. Since f is extension of hk^{-1} , we

have $f(a_2) \in A_1$ for $a_2 \in A_2$ showing K is well defined. Clearly $K(a_2, 0) = r(a_2)$ and $K(a_2, 1) = a_2$. Therefore A is an AHR.

EXAMPLE (3.5). We give an example of a compact Hausdorff space A which is an AHR, but which is not an absolute retract [5] nor an absolute neighborhood retract [5]. Consider the 2-dimensional compact parallelootope Q_2 , namely all (x, y) such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Let A be the compact subset of Q_2 whose points are all (x, y) such that $0 \leq x \leq 1$ and $y = x/n$ where $n = 1, 2, 3, \dots$ plus points $(x, 0)$, $0 \leq x \leq 1$. A not being locally connected at points $(x, 0)$, $0 < x \leq 1$, implies A is not absolute retract or absolute neighborhood retract [5, Corollary 8.3]. Define mapping $f: Q_2 \rightarrow A$ by $f(Q_2) = (0, 0)$. Clearly $f|_A \simeq i_A$ because of mapping $H: A \times (0, 1) \rightarrow A$ defined by $H(x, y, t) = (tx, ty, 0)$. By (3.4) A is an AHR.

In view of (3.3) and (3.5) the concept AHR is a generalization of the concept of absolute retract.

THEOREM (3.6). *A necessary and sufficient condition for a set A to be an AHR is that A be a compact Hausdorff space and that any mapping f defined on a closed subset P of a compact Hausdorff space P_1 such that $f: P \rightarrow A$, admits an h -extension over P_1 relative to A .*

PROOF. NECESSITY. By (3.4) A is homeomorphic to a closed h -retract, say $h(A) = A_1$ of some compact parallelootope Q . Apply Tietze's extension theorem [4] to mapping $hf: P \rightarrow Q$ and obtain extension g of hf over P_1 relative to Q . Clearly g is also an h -extension of hf over P_1 relative to Q . Since $hf(P) \subset A_1$, apply (2.4) and obtain h -extension k of hf over P_1 relative to A_1 . $h^{-1}k$ is h -extension of f over P_1 relative to A because $k|_P \simeq hf: P \rightarrow A_1$ and $h^{-1}: A_1 \rightarrow A$ implies by (2.3) that $(h^{-1}k)|_P \simeq f: P \rightarrow A$.

SUFFICIENCY. Consider any topological image $h(A) = P$ such that P is subset of compact Hausdorff space P_1 . We will show P is h -retract of P_1 . A is compact implies $h(A) = P$ is closed subset of P_1 . Since $h^{-1}: P \rightarrow A$, there exists by hypothesis an h -extension g of h^{-1} over P_1 relative to A . The mapping $hg: P_1 \rightarrow P$ is an h -retraction of P_1 into P because $g|_P \simeq h^{-1}: P \rightarrow A$ and $h: A \rightarrow P$ imply by (2.3) that $(hg)|_P \simeq i_P$. By (3.1) A is an AHR.

THEOREM (3.7). *If $\{A_\sigma\}$, $\sigma \in \Sigma$, is a collection of AHR, then the topological product $\prod A_\sigma$ is an AHR.*

PROOF. A_σ is compact Hausdorff space for each $\sigma \in \Sigma$ implies $\prod A_\sigma$ is compact Hausdorff space [3]. Consider any closed subset P of a compact Hausdorff space P_1 and any mapping $f: P \rightarrow \prod A_\sigma$. For $p \in P$, $f(p) = \{f_\sigma(p)\}$, where $f_\sigma(p) \in A_\sigma$ for each $\sigma \in \Sigma$. Therefore f_σ is map-

ping $f_\sigma: P \rightarrow A_\sigma$ for each $\sigma \in \Sigma$. By (3.6) (necessity) there exists h -extension g_σ of f_σ over P_1 relative to A_σ for each $\sigma \in \Sigma$. Define mapping $G: P_1 \rightarrow \prod A_\sigma$ by $G(p) = \{g_\sigma(p)\}$, for $p \in P_1$. $g_\sigma|P \simeq f_\sigma: P \rightarrow A_\sigma$ for each $\sigma \in \Sigma$ implies there exists a mapping $h_\sigma: P \times (0, 1) \rightarrow A_\sigma$ for each $\sigma \in \Sigma$ such that $h_\sigma(p, 0) = g_\sigma(p)$ and $h_\sigma(p, 1) = f_\sigma(p)$. Define mapping $H: P \times (0, 1) \rightarrow \prod A_\sigma$ by $H(p, t) = \{h_\sigma(p, t)\}$. Clearly $H(p, 0) = G(p)$ and $H(p, 1) = f(p)$. Therefore $G|P \simeq f: P \rightarrow \prod A_\sigma$. By (3.6) (sufficiency) $\prod A_\sigma$ is an AHR.

4. Characterization theorem for absolute retracts. We shall need the following two known results.

THEOREM (4.1). *A necessary and sufficient condition for a space to be an absolute retract (AR) is that it be homeomorphic to a retract of some compact parallelotope [5, Theorem 3.6].*

THEOREM (4.2) (BORSUK'S THEOREM FOR COMPACT HAUSDORFF SPACES). *Let C be a closed subset of a compact Hausdorff space X , let N be an ANR and let $f \simeq g: C \rightarrow N$ (f and g are homotopic mappings of C into N). Then if there is an extension F of f over X relative to N , there is also an extension G of g over X relative to N with $F \simeq G: X \rightarrow N$ [5, Theorem 6.4].*

We can now prove the following characterization theorem for absolute retracts (compact Hausdorff).

THEOREM (4.3). *A necessary and sufficient condition for a space A to be an absolute retract (AR) is that A be both an absolute neighborhood retract (ANR) and an absolute homotopy retract (AHR).*

PROOF. SUFFICIENCY. Since the space A is an AHR, we apply Theorem (3.4) and obtain a homeomorphism, say $h(A) = B$ where B is a closed h -retract of some compact parallelotope Q . Hence there exists an h -retraction, say $f: Q \rightarrow B$ (into), such that $f|B \simeq i_B: B \rightarrow B$, where the mapping $f|B$ is the restriction of the mapping f to B and i_B is the identity mapping on B . The property of being an ANR is a topological invariant [5, Theorem 3.8] and therefore B is an ANR. Since the mapping f is an extension of the mapping $f|B$ over Q relative to B , by Theorem (4.2) there exists an extension r of i_B over Q relative to B . Clearly the mapping $r(Q) = B$ is a retraction of Q onto B . Hence by Theorem (4.1) we have A is an AR.

NECESSITY. Since the space A is an AR we apply Theorem (4.1), and obtain a homeomorphism, say $h(A) = B$ where B is a retract of some compact parallelotope Q . By [5, Theorem 2.2], it follows that B is a closed subset of Q . Since a retract of any space is clearly both a

neighborhood retract and an h -retract of that space, it follows that B is both a closed neighborhood retract and a closed h -retract of the compact parallelotope Q . Hence by [5, Theorem (3.4)] and Theorem (3.4), we have B is both an ANR and an AHR.

It follows from Theorem (4.3) that any AHR which is not an AR cannot be an ANR. Example (3.5) in this paper is an example of such a space which is not locally connected. It is now natural to ask if there exist locally connected AHR (compact Hausdorff) which are not AR (compact Hausdorff). This question remains open.

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