NOTE ON DEGREES OF PARTIAL FUNCTIONS¹

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The notion of one function's being recursive in another is normally considered only for full functions; but Davis [1, p. 171] has given a definition applicable also to partial functions. For one-argument functions (to which we restrict ourselves for the sake of simplicity) this reads: f is partial recursive in g if there is a completely computable functional² Φ for which $f(x) = \Phi(g, x)$. And here Φ is called completely computable if for some partial recursive h we have

$$\Phi(g, x) = t \leftrightarrow (\exists y)(y^{[1]} \subseteq g \text{ and } h(x, y) = t),$$

$\{i^{[1]}\}$ being an effective enumeration of finite functions. In this note we shall argue that Davis' definition does not do justice to our intuitive idea of relative computability; we shall suggest an alternative definition; and we shall show that on either definition there are degrees which are not degrees of any full function.³ In fact we shall show that in a suitable sense "almost no" degrees are degrees of full functions.⁴

Let γ be any nonrecursive set, and let $g_0(2x) = 0$ for $x \in \gamma$, $g_0(2x+1) = 0$ for $x \notin \gamma$, $g_0(x)$ undefined otherwise. Let c_γ be the characteristic function of γ . Then $c_\gamma(x)$ is 0 if $g_0(2x)$ is 0 and 1 if $g_0(2x+1)$ is 0, so clearly c is effectively computable from g_0 in the intuitive sense. None the less, since g_0 is a restriction of the constant function $n(x) = 0$,

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² We use the notation of [1] except that (1) upper-case Greek letters denote functionals; (2) lower-case Greek letters denote sets (of numbers); (3) upper-case italic letters denote classes (of sets) as well as relations; (4) the complement of a set α is written α' ; (5) sometimes we write $[\Phi]$ for $\lambda x \Phi(f, x)$.

³ This fact was known to Lacombe and Shoenfield as early as 1958; but the use of a category argument to establish it, and the consequent strengthening of the result, is new.

⁴ We say that *almost no semicharacteristic functions* (see below) have a certain property, if the class of all sets whose semicharacteristic functions have this property is of first category; and that *almost no* (weak) *degrees* have a certain property if almost no semicharacteristic functions belong to (weak) degrees having that property. The latter usage is justified by the fact that every weak degree $\mathfrak{D}$ contains a semicharacteristic function (for example c_α^0 where $\alpha = \{2^x \cdot 3^{f(x)} \mid f(x) \text{ is defined}\}$ for any $f \in \mathfrak{D}$). (Not every *strong* degree contains a semicharacteristic function; for example this is not true, by the argument of the following paragraph, for any degree of a full non-recursive function.) Observe finally that there exist functions (for example the function g_0 of the following paragraph) which are not of the same degree as any full function on Davis' definition, though they are on ours.

$\lambda x\Phi(g_0, x)$ is a restriction of the partial recursive function $\lambda x\Phi(n, x)$ for every completely computable Φ : but c_γ , being full and nonrecursive, cannot be a restriction of a partial recursive function. Hence c is not partial recursive in g_0 in the sense of Davis.

We regard f as effectively computable from g in the intuitive sense if there exists a mechanical method by means of which every correct and no incorrect value of f can be computed using only finitely many values of g . If we assume that this method can be formalized in some formal system with recursive rules of inference, we are led to the following amendment of Davis' definition.

f is called partial recursive in g if there is a recursively enumerable relation $R(x, y, t)$ for which

$$(1) \quad f(x) = t \leftrightarrow (\exists y)(y^{(1)} \subseteq g \text{ and } R(x, y, t)).$$

Trivially, if f is partial recursive in g in Davis' sense, it is in ours too. For g full the converse holds (cf. the Corollary of Theorem XIX in [2, p. 331]) but by the immediately preceding counterexample not for g arbitrary. Two functions are called *strongly* (Turing) *equivalent* if each is partial recursive in the other sense of Davis [1, p. 171]; *weakly equivalent* if each is partial recursive in the other in the sense of our definition. If two functions are strongly equivalent they are weakly equivalent; but not conversely by footnote 4 above. The equivalence classes relative to strong (weak) equivalence are called *strong* (weak) *degrees*. Not every strong degree contains a full function (footnote 4). Our main result in this note is that the same is true of weak degrees—in fact (cf. footnote 4) that “almost no” (weak) degree contains a full function. This will be established if we can prove the following theorem.

For each set α , let c_α^0 (the semicharacteristic function of α) be that function which is 0 on α and undefined elsewhere. The class of all sets α for which some full nonrecursive function is partial recursive in c_α^0 in the sense of our definition is of first category.⁶ A fortiori the same is true of the class of all α for which some full nonrecursive function is partial recursive in c_α^0 in Davis' sense, and of the class of all α for which c_α^0 is strongly (weakly) equivalent to some full function.

For the proof, call f partial recursive in g with Gödel-number i if (1) holds where R is the i th recursively enumerable relation in some canonical enumeration. If this is so we write $f = [\Phi; g]$. ($[\Phi; g]$

⁶ We use the topology standard in recursion theory, i.e. we identify sets (or their characteristic functions) with points of $\{0, 1\}^{\aleph_0}$, where $\{0, 1\}$ is given the discrete topology. The collection of all classes $\{\alpha | \beta \subseteq \alpha \text{ and } \alpha \cap \gamma = \emptyset\}$, where β and γ are disjoint finite sets, forms a convenient basis of open classes.

does not exist for all i, g : in fact it exists if and only if

$$y_1^{[1]}, y_2^{[1]} \subseteq g, R_i(x, y_1, t_1), R_i(x, y_2, t_2) \rightarrow t_1 = t_2$$

where R_i is the i th recursively enumerable relation.)

The theorem will follow if we can show that the class of all α for which $[\Phi_i c_\alpha^0]$ is full but not recursive is nowhere dense. Let then N be any basic open class; we seek a subneighborhood N_0 of N such that

$$(2) \quad \alpha \in N_0, [\Phi_i c_\alpha^0] \text{ defined and full} \rightarrow [\Phi_i c_\alpha^0] \text{ recursive.}$$

Let $N = \{\alpha \mid \beta \subseteq \alpha \text{ and } \gamma \cap \alpha = \emptyset\}$. Then N_0 satisfying (2) is defined by cases as follows.

CASE I. $[\Phi_i c_\gamma^0]$ is full. Then set $N_0 = N$. For if α satisfies the hypothesis of (2) we have $\alpha \subseteq \gamma', c_\alpha^0 \subseteq c_\gamma^0, [\Phi_i c_\alpha^0] \subseteq [\Phi_i c_\gamma^0]$. But since $[\Phi_i c_\alpha^0]$ is full, $[\Phi_i c_\alpha^0] = [\Phi_i c_\gamma^0]$ and is therefore recursive.

CASE II. $[\Phi_i c_\gamma^0]$ is defined, but not full. Again set $N_0 = N$. For as in Case I, $\alpha \in N_0 \rightarrow [\Phi_i c_\alpha^0] \subseteq [\Phi_i c_\gamma^0]$. But then $[\Phi_i c_\alpha^0]$ is not full either, and (2) is vacuously true.

CASE III. $[\Phi_i c_\gamma^0]$ undefined. This can only happen if

$$(\exists y_1 y_2 t_1 t_2 x)(y_1^{[1]}, y_2^{[1]} \subseteq c_\gamma^0, R_i(x, y_1, t_1), R_i(x, y_2, t_2), t_1 \neq t_2).$$

Let δ be the union of the domains of $y_1^{[1]}$ and $y_2^{[1]}$ (so that $\delta \subseteq \gamma'$). Then we can set $N_0 = \{\alpha \mid \beta \cup \delta \subseteq \alpha \text{ and } \gamma \cap \alpha = \emptyset\}$. For $\alpha \in N_0 \rightarrow \delta \subseteq \alpha \rightarrow y_1^{[1]}, y_2^{[1]} \subseteq c_\alpha^0 \rightarrow [\Phi_i c_\alpha^0]$ undefined; and again (2) holds vacuously.

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