REFLECTIVE RINGS WITH THE ASCENDING CHAIN CONDITION

E. H. FELLER¹ AND E. W. SWOKOWSKI

0. Introduction. In [2] a ring is defined as *reflective* if it satisfies the condition: an element a is regular in R if and only if $a + N(R)$ is regular in $R/N(R)$, where $N(R)$ denotes the nil radical of R . In [7] R is called *N-quorite* if a regular in R and $n \in N(R)$ implies there exists a' regular in R and $n' \in N(R)$ such that $an' = na'$. In the present paper we prove that a reflective and *N-quorite* ring R which satisfies the ascending chain condition (A.C.C.) for right ideals has a right quotient ring $Q(R)$.² In addition the structure of $Q(R)$ is determined.

This paper extends the work of A. W. Goldie [4] to rings with non-zero nil radical. By using the methods of [2], the theorems proved herein are easy but important consequences of the results given in [4].

1. The quotient ring of a reflective ring. If R denotes a reflective ring which satisfies the A.C.C. for right ideals then $R/N(R)$ is a semi-prime ring³ which satisfies the A.C.C. for right ideals. By Theorem 3.9 of [4], $R/N(R)$ has a regular element and, since R is reflective, R has a regular element. Applying Lemma 1.1* of [7] and Theorem 4.1 of [4] it follows that R has a right quotient ring $Q(R)$. As in the proofs of Statements 1.3 and 1.4 of [2] one can show that $Q(R)$ satisfies the A.C.C. for right ideals and that $N(Q(R)) = \{ab^{-1} \mid a \in N(R), b \text{ regular in } R\}$. Consequently $Q(R/N(R)) \cong Q(R)/N(Q(R))$. This proves that (1) $\Rightarrow$ (2) in

THEOREM 1.1. *If R is a ring which satisfies the A.C.C. for right ideals then the following statements are equivalent.*

1. R is reflective and *N-quorite*.
2. R and $R/N(R)$ have right quotient rings $Q(R)$ and $Q(R/N(R))$ respectively where $Q(R/N(R)) \cong Q(R)/N(Q(R))$.

We now prove that (2) $\Rightarrow$ (1). By 2a of [1] an element a is a unit in $Q(R)$ if and only if $a + N(Q(R))$ is a unit of $Q(R)/N(Q(R))$. Hence an element a is regular in R if and only if $a + N(R)$ is regular in $R/N(R)$.

Received by the editors September 12, 1960.

¹ Research supported by the National Science Foundation under Grant NSF-G13560.

² See [5, p. 118].

³ A ring is semi-prime if it contains no nonzero nilpotent ideals.

LEMMA 1.1. *Suppose a ring R has a right quotient ring $Q(R)$. Then if ab^{-1} is regular in $Q(R)$ it is a unit.*

PROOF. We shall first show that if ab^{-1} is regular in $Q(R)$ then a is regular in R . Suppose $ad=0$, $d \neq 0$ in R . There exist d_1 , b_1 with b_1 regular such that $bd_1=db_1$. Then $0=adb_1=abd_1$ and $(ab^{-1})((bd)b^{-1})=(abd_1)(bb_1)^{-1}=0$, since $b(bd_1)=(bd)b_1$. Suppose $da=0$, $d \neq 0$ in R . There exist a_1 , b_1 with b_1 regular such that $ba_1=ab_1$. Thus $0=dab_1=dba_1$ where $db \neq 0$. Then $((db)b^{-1})(ab^{-1})=(dba_1)(bb_1)^{-1}=0$. Thus if ab^{-1} is regular in $Q(R)$, a is regular in R . Since a is a unit of $Q(R)$ if a is regular in R , then ab^{-1} is a unit of $Q(R)$.

THEOREM 1.2. *Let R denote a reflective and N -quorite ring with the A.C.C. for right ideals and let $Q(R)$ denote the right quotient ring of R . Then*

1. $Q(R)$ is a semi-primary S.B.I. ring.
2. $Q(R)$ is reflective.
3. $Q(R)$ satisfies the A.C.C. for right ideals.

PROOF. By Theorem 4.4 of [4] we know that $Q(R)/N(Q(R))$ is semi-simple and satisfies the descending chain condition for right ideals. Thus the Jacobson radical of $Q(R)/N(Q(R))$ is zero and the Jacobson radical of $Q(R)$ equals $N(Q(R))$. Hence R is semi-primary according to Definition 1 of [6, p. 56]. Since $N(Q(R))$ is a nil ideal we have by Proposition 3 of [6, p. 54] that $Q(R)$ is an S.B.I. ring. (See Definition 1 of [6, p. 53].) We have shown that $Q(R)$ is a semi-primary S.B.I. ring. The fact that $Q(R)$ is reflective follows from Lemma 1.1 and 2a of [1]. To show that $Q(R)$ satisfies the A.C.C. for right ideals we follow the proof of statement 1.3 of [2].

Thus we have determined $Q(R)$ to be a semi-primary S.B.I. ring whose structure is given in [6, pp. 56–57]. Note that $Q(R)$ is a group direct sum of rings, each of which is a total matrix ring over a completely primary ring and a subgroup of the radical.

One might speculate that every ring R which satisfies the A.C.C. for right ideals and where R/N has a nontrivial right quotient ring also has a nontrivial right quotient ring. However, the example given in [2, §1] is a ring which satisfies these conditions yet does not possess a nontrivial right quotient ring since every element is a zero divisor.

REMARK. In [3, p. 594] A. W. Goldie proves that a prime ring with certain chain conditions which is right uniform is an integral domain. The structure of an arbitrary uniform ring with A.C.C. on right ideals can be obtained from [0]. If a ring is uniform then the 0 ideal is intersection irreducible for right ideals. Hence in this case

by [0, Theorem 2.2] the nil radical $N(R) = \{x \mid x^n = 0, x \in R\}$ is completely prime. Thus, if R is semi-prime, uniform and satisfies the A.C.C. for right ideals, then R is an integral domain.

BIBLIOGRAPHY

0. E. H. Feller, *The lattice of submodules of a module over a noncommutative ring*, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 342–357.
1. ———, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. vol. 89 (1958) pp. 79–91.
2. E. H. Feller and E. W. Swokowski, *Reflective N -prime rings with the ascending chain condition*, Trans. Amer. Math. Soc. vol. 99 (1961) pp. 264–271.
3. A. W. Goldie, *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc. vol. 8 (1958) pp. 589–608.
4. ———, *Semi-prime rings with maximum condition*, Proc. London Math. Soc. vol. 10 (1960) pp. 201–220.
5. N. Jacobson, *The theory of rings*, Math. Surveys, vol. 2, 1943.
6. ———, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. 37, 1956.
7. E. H. Feller and E. W. Swokowski, *Correction to [2]*, to appear.

UNIVERSITY OF WISCONSIN, MILWAUKEE, AND
MARQUETTE UNIVERSITY