ON DIFFERENTIABILITY AT THE BOUNDARY IN CONFORMAL MAPPING¹

S. E. WARSCHAWSKI

Introduction. The object of this note is to present a short and simple proof of the following two theorems.

THEOREM 1. Suppose that C is a closed rectifiable Jordan curve in the complex ζ -plane and that C has a continuously turning tangent in a neighborhood of one of its points, ζ_0 . Suppose, furthermore, that the tangent angle $\tau(s)$ as function of the arc length s has a modulus of continuity $\omega(t)$ at the point s_0 which corresponds to ζ_0 , i.e.,

$$|\tau(s) - \tau(s_0)| \leq \omega(|s - s_0|) \quad (|s - s_0| \leq \sigma),$$

where $\omega(t)$ is a nondecreasing continuous function, $0 \leq t \leq \sigma$, and

$$(1) \quad \int_0^\sigma \frac{\omega(t)}{t} dt < \infty.$$

Let $\zeta = f(z)$ map the disk $|z| < 1$ conformally onto the interior of C and let $f(z_0) = \zeta_0$. Then

$$(2) \quad \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists for unrestricted approach in $|z| \leq 1$ ($z \neq z_0$) and

$$\lim_{z \rightarrow z_0} f'(z) = f'(z_0)$$

in any Stolz angle with vertex at z_0 . Furthermore, $f'(z_0) \neq 0$.

THEOREM 2. If $\tau(s)$ is continuous along the entire curve C and has a modulus of continuity $\omega(t)$ along C , where $\omega(t)$ satisfies (1), then $f'(z)$, if defined by (2) on $|z| = 1$, is continuous in the closed disk $|z| \leq 1$. Furthermore, if m denotes an upper bound for $|f'(e^{i\theta})|$ then $\arg f'(e^{i\theta})$ and $\log |f'(e^{i\theta})|$ have the moduli of continuity

$$(3) \quad \omega_0(\theta) = \omega(m\theta) + \theta$$

and

$$(3') \quad \omega_0^*(\theta) = M \left[\int_0^\theta \frac{\omega_0(t)}{t} dt + \theta \int_\theta^\pi \frac{\omega_0(t)}{t^2} dt \right]$$

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respectively, where M is a constant.

Theorem 2 contains as a special case a result due to O. D. Kellogg [2] ($\omega(t) = Kt^\alpha$). Both theorems (except for the statements (3) and (3')) were obtained by the author in [7, p. 443] as special cases of a more general result whose proof was rather lengthy and complicated. Inasmuch as both of these theorems have found applications, a simple direct proof may be of some interest.²

1. **Lemmas.** We present first two lemmas.

LEMMA 1. Suppose that C is a closed rectifiable Jordan curve which possesses the following property at one of its points ζ_0 : there exists a subarc γ , which contains ζ_0 as an interior point, and a constant $c > 1$ such that for any two points ζ_1 and ζ_2 of γ and the corresponding arc Δs of γ between them,

$$(4) \quad \Delta s \leq c |\zeta_1 - \zeta_2|.$$

Suppose that $\zeta = f(z)$ maps the disk $|z| < 1$ conformally onto the interior Ω of C , and that $f(z_0) = \zeta_0$, $z_0 = e^{i\theta_0}$. Then there exists a constant H such that for $|z| \leq 1$

$$(5) \quad |f(z) - f(z_0)| \leq H |z - z_0|^\alpha \quad (\alpha = 2/(1+c)^2).$$

Furthermore, the arc length $s(\theta)$, as function of θ (where $z = e^{i\theta}$) is $\text{Lip}(\alpha)$ at $\theta = \theta_0$, i.e.,

$$(6) \quad |s(\theta) - s(\theta_0)| \leq K |\theta - \theta_0|^\alpha.$$

COROLLARY. If there exists a constant $c > 1$ such that (4) holds uniformly for all ζ_1 and ζ_2 on C , where Δs is the length of the (shorter) arc of C between ζ_1 and ζ_2 , then (5) and (6) hold uniformly for all $z_0 = e^{i\theta_0}$ on the unit circle.

PROOF. Let k_ρ denote the arc of the circle $|z - z_0| = \rho < 1$ which lies in $|z| \leq 1$. We take $\rho \leq a$ where a is so small that the image of the arc $\{z | |z| = 1, |z - z_0| \leq a\}$ is a subarc of γ . Let $\Delta_\rho = \{z | |z - z_0| < \rho, |z| < 1\}$, $0 < \rho \leq a$. We have

$$l_\rho^2 = \left(\int_{k_\rho} |f'(z_0 + \rho e^{i\phi})| \rho d\phi \right)^2 \leq \int_{k_\rho} |f'(z)|^2 \rho d\phi \cdot \pi \rho \quad (z = z_0 + \rho e^{i\phi}),$$

² Even the proofs given for the special case of Kellogg's theorem by Kellogg in [2] in 1912, by this author in [8] in 1932 and by Goluzin [1] in 1952 seem to be less elementary than the one given below. It may well be presented in an introductory course on conformal mapping.

Our proof of Theorem 1 still holds if the continuity of $\tau(s)$ for $|s - s_0| \leq \sigma$ is replaced by the weaker assumption that C has an " L -tangent" at ζ_0 [5, p. 93] and that $|\tau(s) - \tau(s_0)| \leq \omega(|s - s_0|)$ for almost all s in $|s - s_0| \leq \sigma$.

and therefore, for $0 < r \leq a$,

$$\int_0^r \frac{l_\rho^2}{\rho} d\rho \leq \pi \int_0^r \int_{k_\rho} |f'(z)|^2 \rho d\phi d\rho = \pi A(r)$$

where $A(r)$ denotes the area of $f(\Delta_r)$. Because of (4) the length of boundary arc of $f(\Delta_r)$ which is a part of C does not exceed cl_r and therefore we have by the isoperimetric inequality

$$A(r) \leq \frac{(l_r + cl_r)^2}{4\pi} = l_r^2 \frac{(1+c)^2}{4\pi}.$$

Hence,

$$F(r) = \int_0^r \frac{l_\rho^2}{\rho} d\rho \leq l_r^2 \left(\frac{1+c}{2} \right)^2.$$

Since for almost all r , $0 < r \leq a$, $rF'(r) = l_r^2$ we have

$$F(r) \leq \frac{r}{2\alpha} F'(r), \quad \alpha = \frac{2}{(1+c)^2},$$

or

$$\frac{2\alpha}{r} \leq \frac{F'(r)}{F(r)}$$

and hence, for $0 < r \leq a$

$$\frac{F(r)}{r^{2\alpha}} \leq \frac{F(a)}{a^{2\alpha}} \leq \frac{\pi A}{a^{2\alpha}} = B_0$$

where A is the area of Ω . Now there exists for every $r \leq a$ an r_1 with $r/2 \leq r_1 \leq r$ such that

$$l_{r_1}^2 \log 2 \leq F(r) \leq B_0 r^{2\alpha}$$

or

$$l_{r_1} \leq B r^\alpha.$$

Thus, if z is a point in $|z| \leq 1$ with $|z - z_0| = r/2$,

$$|f(z) - f(z_0)| \leq l_{r_1}(1+c) \leq (1+c)B2^\alpha \cdot \left(\frac{r}{2}\right)^\alpha = H|z - z_0|^\alpha.$$

This proves (5). The inequality (6) follows from (5) by use of (4).

In order to verify the corollary we only need to note that we can choose the constant a such that, uniformly for all z_0 on $|z_0| = 1$, the image of the arc $\{z \mid |z - z_0| \leq a, |z| = 1\}$ by means of $f(z)$ is shorter than one half of the length of C . Then the estimates (5) and (6) just derived hold uniformly for $|z_0| = 1$.

LEMMA 2. Suppose C is a closed rectifiable Jordan curve which possesses a continuous tangent in a neighborhood of a point ζ_0 . Let $\zeta = g(z)$ map the half-plane $\text{Im}\{z\} > 0$ conformally onto the interior of C such that $g(0) = \zeta_0$. Then, uniformly for $0 \leq \phi_1 \leq \phi_2 \leq \pi$,

$$(7) \quad \lim_{\rho \rightarrow 0} \left| \frac{g(\rho e^{i\phi_2})}{g(\rho e^{i\phi_1})} \right| = 1.^3$$

PROOF. We may assume without loss of generality that the tangent to C at ζ_0 is the real and that the interior normal is the positive imaginary axis. By a theorem of Lindelöf [3] (see also [4, pp. 126–127; 1, pp. 371–372]) $\arg g'(z)^4$ has continuous boundary values on $y = 0$ in a neighborhood of $z = 0$, and

$$(8) \quad \arg \left(\frac{g(z)}{z} \right) \rightarrow 0 \quad \text{and} \quad \arg g'(z) \rightarrow 0$$

as $z \rightarrow 0$ in $y \geq 0$. For all sufficiently small ρ , say $0 \leq \rho \leq \rho_0$, the curve $\gamma_\rho: \zeta = g(\rho e^{i\phi})$, $0 \leq \phi \leq \pi$ (ρ fixed), has, therefore, continuous tangent. Furthermore, if α denotes the tangent angle and ψ the polar angle at the point $\zeta = g(\rho e^{i\phi})$ we have

$$\alpha - \psi = \arg g'(\rho e^{i\phi}) + \phi + \frac{\pi}{2} - \arg g(\rho e^{i\phi}),$$

and by (8), uniformly for $0 \leq \phi \leq \pi$,

$$\left| \alpha - \psi - \frac{\pi}{2} \right| \leq \delta(\rho), \quad \lim_{\rho \rightarrow 0} \delta(\rho) = 0.$$

This inequality implies that any ray $\arg \zeta = \psi$ intersects γ_ρ in at most one point. Hence, γ_ρ may be represented in polar coordinates in the form $r = r(\psi)$, where $r'(\psi)$ is continuous and

³ The conclusion of this lemma holds under a much weaker assumption on C than that of the continuity of the tangent at ζ_0 (e.g. if C has a tangent at ζ_0 and is "regulär unbewallt" as defined in [7, p. 357]; see J. Wolff [9]). The above formulation may also be obtained from a result of Ostrowski [5, p. 118], but for the sake of completeness we give an independent proof.

⁴ Since $g'(z) \neq 0$ in $y > 0$, $\arg f'(z)$ is single-valued once it is normalized at one point.

$$\left| \frac{r'(\psi)}{r(\psi)} \right| = |\cot(\alpha - \psi)| \leq \tan \delta(\rho).$$

By the mean value theorem we have, therefore, for any ψ_1 and ψ_2 in the interval in which $r(\psi)$ is defined,

$$|\log r(\psi_2) - \log r(\psi_1)| \leq |\psi_2 - \psi_1| \tan \delta(\rho) \leq 2\pi \tan \delta(\rho),$$

since $|\psi_2 - \psi_1| \leq 2\pi$. This inequality implies the conclusion (7).

2. Proof of Theorem 1. Since C has a continuous tangent in a neighborhood of ζ_0 , it follows from Lindelöf's theorem [3] that $\arg f'(z)$ assumes continuous boundary values on an arc β of $|z| = 1$ which contains z_0 as interior point. We normalize $\arg f'(z)$ by requiring that $\arg f'(0) = 0$. The rectifiability of C insures that $\arg f'(z)$ has boundary values, which we denote by $\arg f'(e^{i\theta})$, almost everywhere on $|z| = 1$. Without loss of generality we may make the further assumption that $\arg f'(z)$ may be represented by the Poisson integral in $|z| < 1$ with the boundary function $\arg f'(e^{i\theta})$. Otherwise, we construct a closed Jordan curve Γ with continuous tangent, which has a subarc β' of β , containing z_0 as interior point, in common with $|z| = 1$ and is otherwise contained in $|z| < 1$. We may also assume that $z=0$ is in the interior of Γ . Let $z=h(w)$ map the disk $|w| < 1$ conformally onto the interior of Γ , $h(0)=0$. Then $\arg \{ (d/dw)f[h(w)] \} = \arg f'[h(w)] + \arg h'(w)$ is continuous in $|w| \leq 1$ and is thus representable by the Poisson integral in $|w| < 1$. Since $h^{-1}(z)$ is analytic on β' with a non-vanishing derivative at z_0 , it is sufficient to prove the theorem for $f[h(w)]$ in order to insure its validity for $f(z)$.

Under the assumption that $\arg f'(z)$ is represented by the Poisson integral in $|z| < 1$ the conjugate function $\log |f'(z)|$ approaches

$$(9) \quad h(\theta_0) = \frac{1}{2\pi} \int_0^\pi [\arg f'(z_0 e^{it}) - \arg f'(z_0 e^{-it})] \cot \frac{t}{2} dt + \log |f'(0)|$$

as $z \rightarrow z_0 = e^{i\theta_0}$ in any Stolz angle, if the latter integral converges absolutely [10, p. 54]. This together with the continuity of $\arg f'(z)$ at z_0 will establish (2), with $f'(z_0) = \exp[h(\theta_0) + i \arg f'(e^{i\theta_0})]$.

To prove the absolute convergence of (9) we note that

$$\arg f'(e^{i\theta}) = \tau[s(\theta)] - \theta - \frac{\pi}{2}$$

and therefore

$$\begin{aligned} |\arg f'(e^{i(\theta_0+t)}) - \arg f'(e^{i\theta_0})| &\leq |\tau[s(\theta_0+t)] - \tau[s(\theta_0)]| + |t| \\ &\leq \omega(K|t|^\alpha) + |t| \end{aligned}$$

by (6). Hence, for some δ , $0 < \delta < \pi$,

$$(10) \int_0^\delta |\arg f'(e^{i(\theta_0+t)}) - \arg f'(e^{i\theta_0})| \cot \frac{t}{2} dt \leq 2 \int_0^\delta \frac{\omega(Kt^\alpha)}{t} dt + 2\delta \\ = \frac{2}{\alpha} \int_0^{K\delta^\alpha} \frac{\omega(u)}{u} du + 2\delta.$$

Finally, to prove (1) we note first that the relation (2) implies (1) in any Stolz angle. To obtain (1) also for unrestricted approach we apply Lemma 2. Let c_ρ denote the circle of radius ρ ($0 < \rho < 1$) orthogonal to the circle $|z| = 1$ which contains z_0 in its interior and intersects $|z| = 1$ at points equidistant from z_0 . Then, for any z_1 and z_2 on c_ρ , by Lemma 2,

$$\lim_{\rho \rightarrow 0} \left| \frac{f(z_2) - f(z_0)}{f(z_1) - f(z_0)} \right| = 1,$$

uniformly in z_1, z_2 . Thus, if z_1 is chosen on the radius ending at z_0 and $z = z_2$ is an arbitrary point on c_ρ ,

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \sim \left| \frac{f(z_1) - f(z_0)}{z_1 - z_0} \right| \quad \text{as } \rho \rightarrow 0.$$

But since the second term has the limit $|f'(z_0)| = \exp[h(\theta_0)]$ it follows that

$$(11) \quad \lim_{z \rightarrow z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)|$$

for unrestricted approach in $|z| \leq 1$. Moreover, by Lindelöf [3]

$$(12) \quad \lim_{z \rightarrow z_0} \left[\arg \frac{f(z) - f(z_0)}{z - z_0} \right] = \arg f'(z_0)$$

for unrestricted approach. Hence (11) and (12) imply (1).

3. Proof of Theorem 2. The assumption that the C possesses a continuous tangent throughout assures (1) that the corollary to Lemma 1 holds and (2) that $\arg f'(z)$ has continuous boundary values on $|z| = 1$. Then the argument of the preceding section proving the absolute convergence of the integral (9), shows that it converges uniformly for all θ_0 in $[0, 2\pi]$. Hence $f'(z)$ is continuous in $|z| \leq 1$.

The statement regarding the modulus of continuity of $\arg f'(e^{i\theta})$ follows immediately from the inequalities ($t > 0$)

$$\begin{aligned} |\arg f'(e^{i(\theta_0+t)}) - \arg f'(e^{i\theta_0})| &\leq |\tau[s(\theta_0+t)] - \tau[s(\theta_0)]| + t \\ &\leq \omega(mt) + t = \omega_0(t). \end{aligned}$$

Then the expression given for $\omega_0^*(\theta)$ follows from a theorem of J. L. Walsh and Margaret Elliott [6].

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UNIVERSITY OF MINNESOTA