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A SUBSTITUTE FOR LEBESGUE'S BOUNDED CONVERGENCE THEOREM

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1. Lebesgue's bounded convergence theorem has become a powerful tool in the theory of linear topological spaces, and recently, for a treatment of weak convergence of sequences or for a proof of Krein's theorem, the tendency is to use it in an essential way.¹ The following is a useful substitute for the bounded convergence theorem stated in the language of linear space theory.

THEOREM 1. *Let C be a compact (or countably compact)² subset of a (real or complex) linear topological space E , and let $\{f_n\}$ be a sequence of continuous linear functionals on E which is uniformly bounded on C . If, for each x in C , $\lim_n f_n(x) = 0$, then the same equality holds for every x in the closed convex extension of C .*

In case C is compact and Hausdorff, the proof of Theorem 1 may run as follows: Let F be the Banach space of all scalar-valued continuous functions on C with the supremum norm; then there is a linear transformation T on the dual E^* of E into F defined by the equation $T(f) = f|_C$. Let x_0 be a point in the closed convex extension of C . Then one can define a bounded functional ϕ on the range of T

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¹ I am indebted to the referee for the remark that, in Dunford and Schwartz [2], Krein's theorem is proved using Riesz-Markoff-Kakutani's theorem but not Lebesgue's bounded convergence theorem. Their proof relies on the theory of integration of vector-valued functions.

² A space X is countably compact if each sequence in X has a cluster point.

by requiring that $\phi(T(f)) = f(x_0)$. By the Hahn-Banach extension theorem, ϕ can be extended to a bounded functional $\bar{\phi}$ on F . Hence there exists a finite Baire measure μ on C such that $\phi(g) = \int_C g d\mu$ for each g in F . In particular, for each f in E^* , $f(x_0) = \phi(T(f)) = \int_C T(f) d\mu = \int_C f d\mu$. Since the sequence $\{f_n\}$ is uniformly bounded on C and $\lim_n f_n(x) = 0$ for each x in C , it follows from the bounded convergence theorem that $\lim_n f_n(x_0) = \lim_n \int_C f_n d\mu = 0$.

The purpose of the present note is to give a proof of Theorem 1 without using either the Riesz-Markoff-Kakutani representation of the dual of the space of continuous functions on a compact Hausdorff space or Lebesgue's bounded convergence theorem. Instead, our proof is based on a theorem (Theorem 3) which is combinatorial in nature. Our approach is related to Banach's beautiful exposition of weak convergence in the *annexe* of his book (Banach [1]), because the lemma needed to prove Theorem 3 is contained in the proof of Theorem 5 of Banach [op. cit. p. 219]. As an illustration, a sketch of a proof using Theorem 1 of Krein's theorem on weak compactness is included at the end of the note. Our terminology agrees with that of Kelley [5].

2. A nonvoid family $\mathcal{A}$ of sets is called an *algebra* if the union I of $\mathcal{A}$ belongs to $\mathcal{A}$ and, whenever A and B are members of $\mathcal{A}$, $A \cup B$ and $I \sim A$ (and hence $A \cap B$ and the null set) belong to $\mathcal{A}$. A real-valued function m on $\mathcal{A}$ is called *additive* if for any disjoint members A and B of $\mathcal{A}$ it holds that $m(A \cup B) = m(A) + m(B)$. The proof of the following lemma is included here for the sake of completeness.

LEMMA 2. *Let m be a non-negative real-valued additive function on an algebra of sets $\mathcal{A}$, and let $\{A_k\}$ be a sequence of members of $\mathcal{A}$ such that, for some positive number e , $m(A_k) > e$ for all k . Then there exists a subsequence $\{A_{k_i}\}$ of $\{A_k\}$ such that $m(\cap \{A_{k_i} : i = 1, 2, \dots, n\}) > 0$ for all n .*

PROOF. It is sufficient to prove, under the hypothesis of the lemma, that there exists a subsequence $\{A_{k_j}\}$ of $\{A_k\}$ such that, for some positive number d , $m(A_{k_1} \cap A_{k_j}) > d$ for $j = 2, 3, \dots$, because then the new sequence $\{A_{k_1} \cap A_{k_j} : j = 2, 3, \dots\}$ satisfies the hypothesis of the lemma and the lemma will follow by repeating the process.

Let $B_n = \cup \{A_k : k = 1, 2, \dots, n\}$; then $\{m(B_n)\}$ is a bounded non-decreasing sequence of real numbers. Let $a = \lim_n m(B_n)$. Then there is a positive integer r such that $a - m(B_r) < e/2$. Hence, for any integer k greater than r , $m(A_k \sim B_r) \leq m(B_k \sim B_r) = m(B_k) - m(B_r) < e/2$, so that $m(A_k) > e$ implies that $m(A_k \cap B_r) > e/2$, which in turn

implies that there is an integer s such that $1 \leq s \leq r$ and $m(A_s \cap A_k) \geq e/2r$. Therefore there is an s such that $m(A_s \cap A_k) \geq e/2r$ holds for an infinite number of k 's, and the assertion of the previous paragraph is established. The proof of the lemma is complete.

In the sequel, if A is a finite set, we denote the number of elements in A by $c(A)$.

THEOREM 3. *Let $\{I_i\}$ be a sequence of pairwise-disjoint finite non-void sets and let $I = \bigcup \{I_i: i=1, 2, \dots\}$. If $\{A_k\}$ is a sequence of subsets of I such that, for some positive number e , $\liminf_i c(A_k \cap I_i)/c(I_i) > e$ for each k , then there is a subsequence $\{A_{k_i}\}$ of $\{A_k\}$ such that $\bigcap \{A_{k_i}: i=1, 2, \dots, n\}$ is infinite for each n .*

PROOF. Let $\mathcal{A}$ be the algebra of all subsets of I , and for each i define a function m_i on $\mathcal{A}$ by $m_i(A) = c(A \cap I_i)/c(I_i)$ for an A in $\mathcal{A}$. Then m_i is a non-negative real-valued additive function, and $m_i(A) \leq 1$ for all A . Hence by Tychonoff's theorem (Kelley [5, p. 143]), the sequence of functions $\{m_i: i=1, 2, \dots\}$ has a cluster point, say m , with respect to the topology of pointwise convergence. It is easy to see that m is again a non-negative additive function on $\mathcal{A}$ and that, for each k , $m(A_k) > e$. Hence by Lemma 2, there is a subsequence $\{A_{k_i}\}$ of $\{A_k\}$ such that $m(\bigcap \{A_{k_i}: i=1, 2, \dots, n\}) > 0$ for all n . Since $m(A) > 0$ implies that A is infinite, the theorem is established.

In the proof above, instead of $\mathcal{A}$, we could have used the smallest algebra $\mathcal{A}_0$ containing the sequence $\{A_i: i=1, 2, \dots\}$. Then by Theorem C of Halmos [4, p. 23] $\mathcal{A}_0$ is countable, and the use of Tychonoff's theorem may be replaced by the diagonal process of extracting a converging subsequence (on $\mathcal{A}_0$) of $\{m_i\}$.

In our application of Theorem 3, we use only the fact that, if a sequence $\{A_k\}$ satisfies the condition of Theorem 3, then there is a subsequence $\{A_{k_i}\}$ such that $\bigcap \{A_{k_i}: i=1, 2, \dots, n\}$ is nonempty for each n . However, since this formally weakened form of Theorem 3 immediately implies Theorem 3 as stated, its apparent weakness is illusory.

3. PROOF OF THEOREM 1. Clearly it is sufficient to prove the theorem under the condition that C is countably compact. Without loss of generality we can assume that $\sup\{|f_n(x)|: x \in C\} \leq 1$ for each n . Let $\langle C \rangle$ denote the convex extension of the set C , and suppose that, for some x in $\langle C \rangle^-$, it is false that $\lim_n f_n(x) = 0$. Then, if necessary by taking a subsequence, we can assume that, for some positive number e , $|f_n(x)| > e$ for all n . Now, for each i , there is an x_i in $\langle C \rangle$ such that $|f_n(x_i)| > e$ for each $n \leq i$, since x belongs to the closure of $\langle C \rangle$.

It is possible to choose x_i in the form: $x_i = 1/c(I_i) \sum \{x_a: a \in I_i\}$, where I_i is a finite index set and $x_a \in C$ for each a in I_i . (It may happen that $x_a = x_b$ for different a and b in I_i .) It is also possible to choose index sets so that $I_i \cap I_j$ is void if $i \neq j$. Let $I = \bigcup \{I_i: i = 1, 2, \dots\}$ and $A_k = \{a: a \in I \text{ and } |f_k(x_a)| > e/2\}$. If $i \geq k$, then

$$\begin{aligned} e &< |f_k(x_i)| \leq 1/c(I_i) \sum \{|f_k(x_a)| : a \in I_i\} \\ &= 1/c(I_i) [\sum \{|f_k(x_a)| : a \in A_k \cap I_i\} + \sum \{|f_k(x_a)| : a \in I_i \sim A_k\}] \\ &\leq 1/c(I_i) [c(A_k \cap I_i) + (e/2)c(I_i \sim A_k)] \leq c(A_k \cap I_i)/c(I_i) + e/2. \end{aligned}$$

Hence, $i \geq k$ implies that $c(A_k \cap I_i)/c(I_i) > e/2$. In view of Theorem 3 above, there is a subsequence $\{A_{k_i}\}$ of $\{A_k\}$ such that, for each n , the intersection $\bigcap \{A_{k_i}: i = 1, 2, \dots, n\}$ contains at least one element say a_n . Let $y_n = x_{a_n}$; then y_n is in C , and, for each i such that $i \leq n$, it holds that $|f_{k_i}(y_n)| > e/2$. Let y_0 be a cluster point in C of the sequence $\{y_n\}$. Then for each i , $|f_{k_i}(y_0)| \geq e/2$, but this contradicts $\lim_n f_n(y_0) = 0$ and the proof is complete.

Finally, we give a sketch of a proof of Krein's theorem which asserts that if A is a weakly compact subset of a Banach space E , then the closed convex extension $\langle A \rangle^-$ of A is also weakly compact. By Eberlein's theorem (Eberlein [3]) it suffices to prove that $\langle A \rangle^-$ is countably compact; hence without loss of generality, E can be assumed to be separable. Let e be the evaluation map of E into E^{**} ; then, since A is bounded, the weak* closure $\langle e[A] \rangle^-$ is weak* compact. Let $\{f_n\}$ be an arbitrary sequence in the unit sphere of E^* tending to zero relative to the weak* topology. Regarding $\{f_n\}$ as a sequence of weak* continuous functionals on E^{**} , one concludes from Theorem 1 that $\phi(f_n) \rightarrow 0$ for each ϕ in $\langle e[A] \rangle^-$. Therefore, since the unit sphere of E^* is metrizable (because E is separable), each ϕ in $\langle e[A] \rangle^-$ is weak* continuous on the unit sphere of E^* and hence on the whole of E^* (for E is complete). It follows that the weak* compact set $\langle e[A] \rangle^-$ is contained in $e[E]$ and the theorem of Krein follows in turn.

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