ON THE MAXIMUM MODULUS AND THE MAXIMUM TERM OF AN ENTIRE DIRICHLET SERIES

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1. We consider the Dirichlet series

$$(1) \quad f(s) = \sum_{n=0}^{\infty} a_n e^{\lambda_n s} \quad (s = \sigma + it, \lambda_0 = 0, \lambda_n < \lambda_{n+1} \rightarrow \infty)$$

which we assume to be absolutely convergent for all s . We define the maximum modulus over a vertical line as

$$M(\sigma) = \text{l.u.b.}_{-\infty < t < \infty} |f(\sigma + it)|$$

and the maximum term as

$$m(\sigma) = \max_{n \geq 0} |a_n e^{\lambda_n(\sigma + it)}|.$$

Finally, the index function $n(\sigma)$ is given by

$$n(\sigma) = \max_{n \geq 0} \{n \mid m(\sigma) = |a_n| e^{\lambda_n \sigma}\}.$$

The function $\log M(\sigma)$ is divergent to $+\infty$ for $\sigma \rightarrow +\infty$ [1, p. 77] and convex [2, p. 240]. By analogy with the case of Taylor series, [3, pp. 37-38] and making use of the absolute convergence of (1), it is an easy matter to show that the same properties hold for $m(\sigma)$, and that $n(\sigma)$ is a step function divergent to $+\infty$ for $\sigma \rightarrow +\infty$, continuous from the right for all σ , having only a finite number of discontinuities in every infinite interval of the form $(-\infty, b)$, and related to $m(\sigma)$ by the formula, valid for all σ and σ' ,

$$(2) \quad \log m(\sigma) = \log m(\sigma') + \int_{\sigma'}^{\sigma} \lambda_{n(x)} dx.$$

It is also known [4, p. 170, Theorem IX] that

$$(3) \quad m(\sigma) \leq M(\sigma).$$

If we denote by n_0 the index of the first a_n which is not zero and by $\{\sigma_i\}$ ($i=0, 1, 2, \dots$) the monotonic increasing sequence of discontinuities of $n(\sigma)$ (taking $\sigma_0 = -\infty$ for the sake of completeness) then

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the sequence of indices defined by $n_i = n(\sigma_i)$ ($i=0, 1, 2, \dots$) is such that

$$(4) \quad n(\sigma) = n_i \quad \text{for } \sigma_i \leq \sigma < \sigma_{i+1}.$$

2. Since the assumption that the series (1) is absolutely convergent for all s implies $\log |a_n|/\lambda_n \rightarrow -\infty$, it is possible to construct a Newton-Hadamard polygon for the sequence of points P_n with coordinates $\lambda = \lambda_n$, $y = -\log |a_n|$ in a cartesian plane (λ, y) . The sequence of vertices of such polygon corresponds to the sequence of indices n_i and has the property that the slope of the side $P_{n_{i-1}}P_{n_i}$ is equal to σ_i . The proofs of these facts are entirely similar to the ones in the case of Taylor series [5, p. 6; 6, p. 274].

3. Let us now assume that three strictly increasing, divergent sequences $\{n_i\}$, $\{\sigma_i\}$ and $\{\lambda_n\}$ ($i, n=0, 1, 2, \dots$; $\lambda_0=0$, $\sigma_0=-\infty$) with the n_i being non-negative integers, and the λ 's such that

$$(5) \quad \infty \geq \liminf_{n \rightarrow \infty} \lambda_n / \log n = \gamma > 0$$

are given, and let us consider the series

$$(6) \quad \sum_{n=0}^{\infty} A_n e^{\lambda_n s}$$

where the A_n are defined in the following way: $A_n=0$ for $n < n_0$, $A_{n_0}=a$, a being any arbitrary real constant, and $A_n = A_{n_{i-1}} \exp[-(\lambda_n - \lambda_{n_{i-1}})\sigma_i]$ for $n_{i-1} < n \leq n_i$. Since the sequence $\{-\log(A_n/A_{n-1})/(\lambda_n - \lambda_{n-1})\}$ is obviously monotonic and divergent, it follows that the sequence of points Q_n of the plane (λ, y) having coordinates $\lambda = \lambda_n$, $y = -\log A_n$ defines a Newton-Hadamard polygon whose vertices are Q_{n_i} and whose sides contain all the points Q_n . This implies that the $m(\sigma)$ function corresponding to the series (6) is given by $m(\sigma) = A_{n_i} \exp(\lambda_{n_i}\sigma)$ for $\sigma_i \leq \sigma < \sigma_{i+1}$. Finally, we can prove that (6) is absolutely convergent for all σ in the following way: For every given $s = \sigma + it$ we can select three numbers γ_1 , γ_2 , K such that $0 < \gamma_1 < \gamma_2 < \gamma \leq \infty$ and $K > \sigma + (1/\gamma_1)$. Then by (5) we have that from some n on, $\lambda_n > \gamma_2 \log n$. Also, since $\sigma_i \uparrow \infty$, it follows that $\log A_n < -K\lambda_n$ for all n large enough. Therefore, there exists some integer N such that

$$\begin{aligned} \sum_{n=N}^{\infty} |A_n e^{\lambda_n s}| &< \sum_{n=N}^{\infty} e^{(\sigma-K)\lambda_n} < \sum_{n=N}^{\infty} e^{-(1/\gamma_1)\lambda_n} < \sum_{n=N}^{\infty} e^{-(\gamma_2/\gamma_1) \log n} \\ &= \sum_{n=N}^{\infty} n^{-(\gamma_2/\gamma_1)} < \infty. \end{aligned}$$

Consequently, the series (6) defines an entire function $F(s)$ and if the sequences $\{n_i\}$, $\{\sigma_i\}$, $\{\lambda_n\}$ are the ones corresponding to the series (1), and we take $a = |a_{n_0}|$, then both series (1) and (6) have the same index function $n(\sigma)$ and the same maximum term $m(\sigma)$ given by (2). In addition, their coefficients satisfy the relations $|a_n| \leq A_n$ for all $n=0, 1, 2, \dots$ and $|a_{n_i}| = A_{n_i}$ for all $i=0, 1, 2, \dots$, and finally

$$(7) \quad M(\sigma) \leq \max_{-\infty < t < \infty} |F(\sigma + it)| = F(\sigma)$$

holds for all σ .

4. We now consider the problem of the asymptotic equivalence of the functions $\log M(\sigma)$ and $\log m(\sigma)$. If $\liminf_{n \rightarrow \infty} \log \lambda_n / \log n > 0$, (which implies (5)), holds, and either

$$\limsup_{\sigma \rightarrow \infty} \log \log M(\sigma) / \sigma < \infty$$

or, what is equivalent, [7, p. 109, Theorem (2.8)],

$$\limsup_{\sigma \rightarrow \infty} \log \log m(\sigma) / \sigma < \infty$$

that is to say, if the function is of finite Ritt order [1, p. 77], then $\log M(\sigma)$ and $\log m(\sigma)$ are asymptotically equivalent, [6, p. 265, Theorem 5].

With no restriction on the order but assuming

$$(8) \quad \liminf_{n \rightarrow \infty} (\lambda_n - \lambda_{n-1}) = \beta > 0$$

we will establish sufficient conditions for $\log M(\sigma)$ and $\log m(\sigma)$ to be asymptotically equivalent. We begin by proving the following theorem which generalises the inequality of Valiron [8, p. 32, Theorem 11].

THEOREM 1. *If the series (1) is absolutely convergent for all s and (8) holds, then for every $\beta_1 < \beta$ and all $\sigma > \sigma(\beta_1)$*

$$m(\sigma) \leq M(\sigma) \leq \left[1 + n(\sigma + \delta(\sigma)) + \frac{1}{e^{\beta_1 \delta(\sigma)} - 1} \right] m(\sigma)$$

where $\delta(\sigma)$ is any positive, but otherwise arbitrary, function of σ .

PROOF. For every σ , we define the integer k by the equality $n_k = n(\sigma + \delta(\sigma))$, and for every index $m > n_k$ we define the integer p by $n_p \geq m > n_{p-1}$. It follows that $\sigma + \delta(\sigma) < \sigma_{k+1}$ and $k < p$ and therefore, since (8) implies (5) and the conclusions of the previous section are valid, we have

$$\begin{aligned}
A_m \exp(\lambda_m \sigma) &= a \exp[-(\lambda_{n_1} - \lambda_{n_0})\sigma_1 - \dots - (\lambda_{n_k} - \lambda_{n_{k-1}})\sigma_k - \dots \\
&\quad - (\lambda_m - \lambda_{n_{p-1}})\sigma_p + \lambda_m \sigma] \\
&= A_{n_k} \exp(\lambda_{n_k} \sigma) \exp[-(\lambda_{n_{k+1}} - \lambda_{n_k})\sigma_{k+1} - \dots \\
&\quad - (\lambda_m - \lambda_{n_{p-1}})\sigma_p + (\lambda_m - \lambda_{n_k})\sigma] \\
&< m(\sigma) \exp[(\lambda_m - \lambda_{n_k})(\sigma - \sigma_{k+1})].
\end{aligned}$$

Now, according to (7), we have for all $\sigma > \sigma(\beta_1)$

$$\begin{aligned}
M(\sigma) &\leq F(\sigma) = \sum_{m=0}^{n_k} A_m \exp(\lambda_m \sigma) + \sum_{m=n_k+1}^{\infty} A_m \exp(\lambda_m \sigma) \\
&< \left\{ 1 + n_k + \sum_{m=n_k+1}^{\infty} \exp[(\lambda_m - \lambda_{n_k})(\sigma - \sigma_{k+1})] \right\} m(\sigma) \\
&\leq \left\{ 1 + n_k + \sum_{m=n_k+1}^{\infty} \exp[(m - n_k)\beta_1(\sigma - \sigma_{k+1})] \right\} m(\sigma) \\
&= (1 + n_k + \exp[\beta_1(\sigma - \sigma_{k+1})]) / \{1 - \exp[\beta_1(\sigma - \sigma_{k+1})]\} m(\sigma) \\
&\leq \{1 + n(\sigma + \delta(\sigma)) + 1/[\exp(\beta_1 \delta(\sigma)) - 1]\} m(\sigma)
\end{aligned}$$

and the theorem is proved.

Finally, we will demonstrate the following result:

THEOREM 2. *Under the same assumptions of Theorem 1, if there is a positive constant α such that either*

$$(9) \quad \lim_{\sigma \rightarrow \infty} \log \log m(\sigma + \alpha) / \log m(\sigma) = 0$$

or

$$(10) \quad \lim_{\sigma \rightarrow \infty} \log \log M(\sigma + \alpha) / \log M(\sigma) = 0$$

then

$$\lim_{\sigma \rightarrow \infty} \log M(\sigma) / \log m(\sigma) = 1.$$

PROOF. By (2) and (8), we have, for all σ large enough and any $\alpha > 0$ and some positive $\beta_1 < \beta$:

$$\begin{aligned}
\log m(\sigma + \alpha) &= \log m(\sigma + \alpha/2) + \int_{\sigma+\alpha/2}^{\sigma+\alpha} \lambda_n(x) dx \geq (\alpha/2) \lambda_n(\sigma + \alpha/2) \\
&\geq (\alpha/2) \beta_1 n(\sigma + \alpha/2).
\end{aligned}$$

Therefore, applying Theorem 1 with $\delta(\sigma) = \alpha/2$, we obtain

$$M(\sigma) \leq \{1 + 2(\alpha\beta_1)^{-1} \log m(\sigma + \alpha) + 1/[\exp(\beta_1 \alpha/2) - 1]\} m(\sigma)$$

and consequently for every $\epsilon > 0$ there exists a number $\sigma(\epsilon)$ such that $\sigma \geq \sigma(\epsilon)$ implies

$$(11) \quad \log M(\sigma) \leq (1 + \epsilon) \log \log m(\sigma + \alpha) + \log m(\sigma)$$

and obviously

$$(12) \quad \log M(\sigma) \leq (1 + \epsilon) \log \log M(\sigma + \alpha) + \log m(\sigma).$$

By division of (11) or (12) by $\log m(\sigma)$ or by $\log M(\sigma)$, and using (9) or (10), respectively, we obtain

$$\log M(\sigma) / \log m(\sigma) \leq o(\sigma) + 1$$

or

$$1 \leq o(\sigma) + \log m(\sigma) / \log M(\sigma).$$

Every one of these inequalities combined with (3) implies

$$\lim_{\sigma \rightarrow \infty} \log M(\sigma) / \log m(\sigma) = 1.$$

Theorem 2 generalises the result given by San Juan in the case of Taylor series [9, p. 134].

An obvious consequence of Theorem 2 is that if (8) holds, then conditions (9) and (10) are equivalent.

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