THE HAAR PROBLEM IN L_1

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Let B be a real Banach space and $E = [x_1, x_2, \cdots, x_n]$ a finite subset of B. Let D be the linear manifold spanned by E. For every $x \in B$ the distance of x from D is attained at some member of D, i.e., there exist scalars $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ such that

$$(1) \qquad \left\| x - \sum_i \alpha_i x_i \right\| = \inf_{y \in D} \| x - y \|.$$

E is said to have the *Haar Property* if the α's in (1) are uniquely determined for every $x \in B$. The *Haar Problem* consists of finding a necessary and sufficient condition on E in order that E have the Haar property.

Haar [1] solved the above problem for the case where B is the collection of real-valued continuous functions on a compact subset of Euclidean n-space under the sup norm. It is easy to see [2] that if B is *strictly-normed*, i.e., if

$$(2) \qquad (\|x + y\| = \|x\| + \|y\|) \Rightarrow y = sx$$

for a scalar s then every linearly-independent set E enjoys the Haar property. Other results have also been obtained [**3**; **4**].

The purpose of this note is to treat the case where B is $L_1(M)$, the collection of real-valued integrable functions on a finite nonatomic measure space M under the L_1 norm. (For $L_p(M)$ with $p > 1$ (2) holds.) The following theorem will be proved:

THEOREM. *No finite subset of $L_1(M)$ has the Haar property.*

The proof depends on the following theorem of Liapounoff and Halmos [**5**; **6**]: *The range of a countably-additive, finite, nonatomic measure with values in a real finite-dimensional vector space is convex.*

LEMMA. *Let $f_1, f_2, \cdots, f_n$ be n functions in $L_1(M)$, M a finite nonatomic measure space. There exists a measurable subset E of M such that $\phi \equiv \chi_E - \chi_{E'}$ satisfies*

$$(3) \qquad \int \phi f_i = 0 \qquad (i = 1, 2, \cdots, n).$$

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PROOF. Assume first that each f_i is either strictly positive, strictly negative, or zero, a.e. in M. Eliminate the zero functions (since for these any ϕ will do), and define a vector-valued measure on the measurable subsets of M by

$$\lambda(E) = \left(a_1 \int \chi_E f_1, a_2 \int \chi_E f_2, \cdots, a_n \int \chi_E f_n\right) \tag{4}$$

where $a_i = (\int f_i)^{-1}$. Since M is nonatomic, each coordinate of λ is nonatomic. By Liapounoff's theorem the range of λ is convex. Since $\lambda(M) = (1, 1, \cdots, 1)$ and $\lambda(0) = (0, 0, \cdots, 0)$, there is a set E in M such that $\lambda(E) = (1/2, 1/2, \cdots, 1/2)$. This E is the E of the lemma.

In the more general case M may be partitioned into a finite disjoint union of measurable sets M_j on any one of which each f_i is strictly positive, strictly negative, or zero. On each M_j the above argument supplies an E_j, and the union of these E_j is the E of the lemma.

PROOF OF THE THEOREM. Let $f_1, f_2, \cdots, f_n$ be n members of $L_1(M)$. With ϕ as in the lemma, define a function g in $L_1(M)$ by $g \equiv \phi[|f_1| + |f_2| + \cdots + |f_n|]$. For any scalars $\{\alpha_i\}$

$$\left\| g - \sum_i \alpha_i f_i \right\| = \int |g - \sum| \geqq \int \phi(g - \sum) = \int \phi g = \|g\|.$$

On the other hand, for every set of scalars $\{\alpha_i\}$ there is an $\epsilon > 0$ such that $|\epsilon \sum \alpha_i f_i| < |g|$ hence

$$\|g - \epsilon \sum \alpha_i f_i\| = \int |g - \epsilon \sum| = \int \phi(g - \epsilon \sum) = \|g\|.$$

Thus for g the α_i in (1) are not only not unique—the admissible vectors $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ contain a sphere about the origin.

Added in Proof: The author has learned that the result given here had already appeared as Theorem 2.5 in [7].

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THE GEÖCZE k-AREA AND A CYLINDRICAL PROPERTY

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In [5], a definition of the Geöcze k-area of a mapping from admissible sets of Euclidean k-space E_k into Euclidean n-space E_n $(2 \leqq k \leqq n)$ is given. This definition is an extension of the Geöcze area given in [3]. With this definition of Geöcze k-area, a treatment of Geöcze k-area is developed for flat mappings $(k=n)$ paralleling the treatment of Geöcze area for plane mappings given in [3]. The present paper gives results concerning the Geöcze k-area for mappings from admissible sets of E_k into E_n $(n>k\geqq 2)$. A cylindrical property is defined for mappings in harmony with [3, (16.10)]. This property, which has had an essential part in the proofs of the main theorems for Lebesgue area for mappings from admissible sets of E_2 into E_n [3] and which has been used in other research, is shown to play a prominent role in the extension of the theory of Geöcze area to higher dimensions. An example is given to show that the theorems concerning the cylindrical property in [3] are no longer valid for $k \geqq 3$. These theorems are shown to be valid under a certain restrictive hypothesis found in the literature.

1. Notations and definitions.

If X is a set in E_k, then $\overline{X}$, X^0, and X^* will denote respectively the closure, interior, and boundary of X.

A *polyhedral region* R in E_k is the point-set covered by a strongly connected k-complex situated in E_k. A polyhedral region R is called *simple* if $E_k - R$ is connected (see [5]).

By a *figure* F we mean a finite union of nonoverlapping polyhedral regions in E_k such that the interior of the union is the union of the interior of the finitely many polyhedral regions. A set A in E_k is said to be *admissible* in each of the following cases:

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