REFERENCES

1. S. Aljančić, R. Bojanić, and M. Tomić, *Sur la valeur asymptotique d'une classe des intégrales définies*, Acad. Serbe Sci. Publ. Inst. Math. vol. 7 (1954) pp. 81–94.
2. E. B. Dynkin, *Limit theorems for sums of independent random quantities*, Izv. Akad. Nauk SSSR vol. 19 (1955) pp. 247–266 (Russian).
3. E. J. Gumbel, *Statistics of extremes*, New York, Columbia University Press, 1958.
4. J. Karamata, *Sur une mode de croissance régulière des fonctions*, Mathematica (Cluj) vol. 4 (1930) pp. 38–53.
5. S. Karlin and J. McGregor, *Occupation time laws for birth and death processes*, Technical Report No. 24, Contract Nonr-225(28), Stanford University, July 29, 1960.
6. J. Lamperti, *Some limit theorems for stochastic processes*, J. Math. Mech. vol. 7 (1958) pp. 433–450.
7. ———, *An occupation time theorem for a class of stochastic processes*, Trans. Amer. Math. Soc. vol. 88 (1958) pp. 380–387.
8. W. L. Smith, *Asymptotic renewal theorems*, Proc. Roy. Soc. Edinburgh. Sect. A vol. 64 (1954) pp. 9–48.

STANFORD UNIVERSITY

ON THE INTERTWINING NUMBER THEOREM

ADAM KLEPPNER

Let G_1 and G_2 be open subgroups of the separable locally compact group $\mathfrak{G}$ and let L and M be unitary representations of G_1 and G_2 respectively. For each x and y in G the indices of the subgroup $H_{x,y} = x^{-1}G_1x \cap y^{-1}G_2y$ in $x^{-1}G_1x$ and in $y^{-1}G_2y$ depend only on the $G_1:G_2$ double coset to which xy^{-1} belongs. Let $\mathfrak{D}_f$ denote the set of all $G_1:G_2$ double cosets for which both these indices are finite and let $\mathfrak{D}$ denote the set of all $G_1:G_2$ double cosets. Let $\mathfrak{I}(L, M; x, y)$ denote the intertwining number of the representations $s \rightarrow L_{xxs}^{-1} = L_s^x$ and $s \rightarrow M_{yxy}^{-1} = M_s^y$ of $H_{x,y}$. The Intertwining Number Theorem of Mackey [1, Theorem 3'] states the following:

- (1) $\mathfrak{I}(L, M; x, y)$ depends only on the $G_1:G_2$ double coset D to which xy^{-1} belongs and may be denoted by $\mathfrak{I}(L, M; D)$;
- (2)
$$\sum_{D \in \mathfrak{D}_f} \mathfrak{I}(L, M; D) \leq \mathfrak{I}(U^L, U^M) \leq \sum_{D \in \mathfrak{D}} \mathfrak{I}(L, M; D);$$
- (3) if L and M are 1-dimensional then $\mathfrak{I}(U^L, U^M) = \sum_{D \in \mathfrak{D}_f} \mathfrak{I}(L, M; D)$.

Received by the editors May 23, 1960.

In his paper [1] Mackey raised the question of finding conditions under which the inequalities of (2) could be asserted to be equalities. It is the purpose of this note to show that if L and M are both finite dimensional then $\mathfrak{I}(U^L, U^M) = \sum_{D \in \mathfrak{D}_f} \mathfrak{I}(L, M; D)$.

For each $y \in G$ and $w \in \mathfrak{H}(M)$ let $g_{y,w}$ be the function: $g_{y,w}(\eta y) = M_\eta(w)$, $\eta \in G_2$, and $g_{y,w}(z) = 0$, $z \notin G_2 y$. Each $g_{y,w} \in \mathfrak{H}(U^M)$ and every function in $\mathfrak{H}(U^M)$ is a sum of functions $g_{y,w}$. Define analogously functions $f_{x,v} \in \mathfrak{H}(U^L)$ for each $x \in G$ and $v \in \mathfrak{H}(L)$. If T is any intertwining operator for the representations U^L and U^M then for all $y \in \mathfrak{G}$ and $w \in \mathfrak{H}(M)$

$$(*) \quad T g_{y,w} = \sum_{\mathfrak{G}/G_1}^x f_{x, A_{x,y}(w)},$$

where $A_{x,y}: \mathfrak{H}(M) \rightarrow \mathfrak{H}(L)$ is a bounded linear operator and the sum is taken over any set of right coset representatives of G_1 in $\mathfrak{G}$. Moreover

$$(**) \quad T^* f_{x,v} = \sum_{\mathfrak{G}/G_2}^y g_{y, A_{x,y}^*(v)},$$

where T^* and $A_{x,y}^* = (A_{x,y})^*$ are the adjoint operators. The function $x, y \rightarrow A_{x,y}$ has the properties (a), (b), (c) of Lemma A of [1], namely:

$$(a) \quad A_{\xi x s, \eta y s} = L_\xi A_{x,y} M_\eta^*, \quad \text{for all } \xi \in G_1, \eta \in G_2, s \in \mathfrak{G},$$

(b) there exists a constant $K > 0$ such that

$$\sum_{\mathfrak{G}/G_1}^x \|A_{x,y}(w)\|^2 \leq K \|w\|^2, \quad \text{for all } y \in \mathfrak{G} \text{ and } w \in \mathfrak{H}(M),$$

(c) there exists a constant $K' > 0$ such that

$$\sum_{\mathfrak{G}/G_2}^y \|A_{x,y}^*(v)\|^2 \leq K' \|v\|^2, \quad \text{for all } x \in \mathfrak{G} \text{ and } v \in \mathfrak{H}(L).$$

Because of (a) the value of A at x, y uniquely determines its values on the double coset $G_1 x y^{-1} G_2$. Also by (a) the numbers $\|A_{x,y}(w)\|$ and $\|A_{x,y}^*(v)\|$ depend only on the G_1 coset to which x belongs and the G_2 coset to which y belongs so that the sums in (b) and (c) are well defined. Each $A_{x,y}$ is an intertwining operator for the representations L^x and M^y of $H_{x,y}$ [1, Lemma D]. Conversely, let A be an intertwining operator for L^x and M^y . Set $A_{\xi x s, \eta y s} = L_\xi A M_\eta^*$ for $\xi \in G_1, \eta \in G_2, s \in \mathfrak{G}$, and $A_{x', y'} = 0$ for $x' y'^{-1} \notin G_1 x y^{-1} G_2$. Defining an operator T by (*) yields a possibly unbounded intertwining operator for U^L and U^M . It is this correspondence between operator-valued functions

$A_{x,y}$ and intertwining operators T which yields the intertwining number theorem. The problem of showing that

$$\mathfrak{I}(U^L, U^M) = \sum_{D \in \mathfrak{D}_f} \mathfrak{I}(L, M; D)$$

reduces to showing that if $x, y \rightarrow A_{x,y}$ is defined by a bounded intertwining operator as in (*) and if $A_{x,y} \neq 0$ for some particular x, y then the double coset $D = G_1 x y^{-1} G_2 \in \mathfrak{D}_f$.

Let $\{r_j\}$ be a set of right coset representatives of $H_{x,y}$ in $y^{-1}G_2y$. Then $G_1 x y^{-1} G_2$ is a disjoint union of cosets $G_1 x r_j y^{-1}$. It follows from (a) that $A_{x r_j y^{-1}, e} = A_{x, y r_j^{-1} y^{-1} y} = A_{x, y} M_{r_j}^y$ and from (b) that

$$(***) \quad \sum_j \|A_{x, y} M_{r_j}^y(w)\|^2 \leq K \|w\|^2, \quad \text{all } w \in \mathfrak{S}(M).$$

Suppose that M is finite dimensional and assume that $[y^{-1}G_2y: H_{x,y}] = \infty$, i.e. that the set $\{r_j\}$ is infinite. Let N be the null space of $A_{x,y}$ and let $w_1, \dots, w_m$ be an orthonormal basis of $\mathfrak{S}(M)$. Because the unit ball in $\mathfrak{S}(M)$ is compact there exists a subsequence $\{r_{j_k}\}$ such that $M_{r_{j_k}}^y(w_i)$ converges to w'_i for each $i=1, \dots, m$. $w'_1, \dots, w'_m$ is also an orthonormal basis of $\mathfrak{S}(M)$ and because $A_{x,y} \neq 0$ there exists an index i such that $w'_i \notin N$. Hence $\|A_{x,y}(w'_i)\| > c > 0$. There exists $d > 0$ such that if $\|w - w'_i\| < d$ then $\|A_{x,y}(w)\| > c$. For k sufficiently large $\|M_{r_{j_k}}^y(w_i) - w'_i\| < d$; hence $\|A_{x,y} M_{r_{j_k}}^y(w_i)\| > c$ for k sufficiently large. But this contradicts (***) and the set $\{r_j\}$ must be finite. A similar argument applied to $A_{x,y}^*$ shows that $[x^{-1}G_1x: H_{x,y}] < \infty$ when L is finite dimensional. Thus $D = G_1 x y^{-1} G_2 \in \mathfrak{D}_f$.

REFERENCE

1. G. W. Mackey, *On induced representations of groups*, Amer. J. Math. vol. 73 (1951) pp. 576-592.

HARVARD UNIVERSITY