FREE TOPOLOGICAL GROUPS

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0. Introduction. In [3; 6; 11] the basic questions of the existence and of the uniqueness of free topological and free topological abelian groups are discussed. Below we present a different approach to these questions and then indicate the consequences of this approach. Whereas Markov [11] resorts to norms and multinorms, Graev [3] to metrics and Kakutani [6] to topological products of large families of groups, we use very special groups of homomorphisms. Inasmuch as our construction may be viewed as a topological product, retopologized, our ideas are most closely related to those of Kakutani.

1. Free topological groups. If X is a completely regular space [7], the free topological group $F(X)$ is defined as a topological group such that:

- (i) X is topologically embeddable in $F(X)$;
- (ii) When embedded as in (i), X generates $F(X)$;
- (iii) If ϕ is a continuous mapping of X into any topological group G , there is an extension Φ of ϕ from X to $F(X)$ such that Φ is a continuous homomorphism of $F(X)$ into G .

Let $F_0(X)$ be the free group generated by the elements of X serving as indeterminates. Thus $F_0(X)$ is the totality of finite words $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$, $\epsilon_i = \pm 1$, $x_i \in X$ with the usual convention for equality, triviality and multiplication. If the family $\mathfrak{F}$ of group topologies on $F_0(X)$ for which X is naturally embedded topologically is nonempty, then [3] the supremum of $\mathfrak{F}$ is the topology that makes $F_0(X)$ into a group $F(X)$ satisfying (i)–(iii). Thus the question of existence turns on the nonemptiness of $\mathfrak{F}$. (The question of uniqueness is settled at once in terms of (iii).)

We proceed as follows: Let $Q^*(X)$ be the set of bounded continuous quaternion-valued functions that are never zero on X . Let $\mathfrak{Q}$ be the group of elements in $Q^*(X)$ that have reciprocals in $Q^*(X)$, i.e., $\mathfrak{Q}$ is the set of elements invertible in the multiplicative structure of $Q^*(X)$. For each x in X define the homomorphism $x: \mathfrak{Q} \rightarrow Q^*$ (the multiplicative group of nonzero quaternions) by the formula $x(f) = f(x)$ for $f \in \mathfrak{Q}$. Let $H = \text{Hom}_c(\mathfrak{Q}, Q^*)$ (continuous homomorphisms

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only) and let H be given the weak topology as follows: A typical neighborhood of the identity e in H is a set

$$U = \{h \mid |h(f_i) - 1| < \epsilon, i = 1, 2, \dots, n\}$$

where $\epsilon > 0$, $f_i \in \mathcal{Q}$ and $|\dots|$ denotes the quaternionic norm. It is easily verified that in this topology H is a topological group. Furthermore, since X is completely regular, distinct points of X correspond to distinct members of H . Hence we regard X as a subset of H and prove

THEOREM 1. *X is contained topologically in H .*

PROOF. A typical neighborhood of an $x_0 \in X$ is a set $U = \{h \mid |h(f_i)x_0^{-1}(f_i) - 1| < \epsilon, i = 1, 2, \dots, n\}$. But $V = \{x \mid |f_i(x)(f_i(x_0))^{-1} - 1| < \epsilon, i = 1, 2, \dots, n\} \subset U$, whence the embedding is continuous. On the other hand, if V is a neighborhood of x_0 in X there is an $f \in \mathcal{Q}$ such that $f(x_0) = 3$, $f(X \setminus V) = 2$, $2 \leq f(X) \leq 3$. Then $U = \{h \mid |h(f) \cdot x_0^{-1}(f) - 1| < 1/4\}$ is such that $U \cap X \subset V$. Hence X is contained topologically in H .

THEOREM 2. *Let H_0 be the least subgroup of H that contains X . Then $F_0(X)$ and H_0 are isomorphic.*

To prove Theorem 2, we establish the following lemmas.

LEMMA 1. *Q^* contains a free subgroup having infinitely many generators.*

PROOF. If R is a rotation in 3-space, then $[1]$, there is a quaternion q such that for each vector $y \equiv (y_1, y_2, y_3)$, $R(y) = z \equiv (z_1, z_2, z_3)$ can be found from the equation $z_1i + z_2j + z_3k = q(y_1i + y_2j + y_3k)q'$ where q' is the conjugate of q . Since $[4; 5; 8]$ the group of rotations in 3-space contains a free group having infinitely many generators $R_1, R_2, \dots$, the corresponding quaternions $q_1, q_2, \dots$ form a system of generators for a free group in Q^* .

LEMMA 2. *Let $p_1, p_2, \dots, p_m$ be distinct points in X . Then there is an $f \in \mathcal{Q}$ such that $f(p_i) = \pm q_i$, $i = 1, 2, \dots, m$ (q_i as in Lemma 1).*

PROOF. Let V_i be a neighborhood of p_i , $V_i \cap V_j = \emptyset$, $i \neq j$. Choose $f_i \in C(X)$ ($\equiv$ set of continuous complex-valued functions on X) so that $0 \leq f_i(X) \leq 1$, $f_i(p_i) = 1$, $f_i(X \setminus V_i) = 0$. Let $W_i = \{x \mid |q_i f_i(x)| > 1/2\} \subset V_i$, and then let $g_i \in C(X)$, $0 \leq g_i(X) \leq 1$, $g_i(X \setminus W_i) = 0$, $g_i(p_i) = 1$. Let $g_0 = \prod_i (1 - g_i)$ and let r_i be the real part (first component) of q_i , $\eta_i = \text{sgn } r_i$. Setting $f = g_0 + \sum_i \eta_i q_i f_i$, we see that $f(p_i) = \pm q_i$. Furthermore, we show $|f|$ is bounded away from 0: In $X \setminus \bigcup_i V_i$, $f = 1$;

in any V_i , $f = 1 - g_i + \eta_i q_i f_i$; in W_i , $|f| > 1/2$; in $V_i \setminus W_i$, $1 - g_i = 1$ and thus $|f| \geq 1$. Since f is continuous and $|f|$ is bounded, $f \in \mathfrak{Q}$.

We can prove Theorem 2 by showing that only trivial products $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ can reduce to the identity in H . Thus let $p_1, p_2, \dots, p_m$ be the distinct points among $x_1, x_2, \dots, x_n$, $m \leq n$. Then for the function f of Lemma 2, clearly $(x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n})(f) \neq 1$ unless the product $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ is trivial.

Thus we have shown that $F_0(X)$ is isomorphic to the subgroup H_0 . On the other hand H_0 contains X topologically. Thus the family $\mathfrak{F}$ is nonempty and the existence of the free topological group is proved. Note that H_0 is a subgroup of the product $\mathbf{P}_x Q_x^*$ where each $Q_x^* = Q^*$ and $x \in X$. However, H_0 may require retopologization (strengthening) to convert it to $F(X)$.

To construct $A(X)$, the free topological abelian group generated by X , we need only replace Q^* by R^+ (the multiplicative group of positive real numbers) and imitate the process described above. There is a curious by-play here: $A(X)$ is the abelianization of $F(X)$ [3], while R^+ is the abelianization of Q^* . We shall discuss this point at greater length below.

2. Applications.

THEOREM 3. X is closed in $F(X)$ (or $A(X)$).

PROOF. We offer the proof only for $F(X)$. If T_0 denotes the topology that H_0 inherits from H , then we show in fact that X is closed relative to T_0 , (a fortiori in $F(X)$). Let $h_0 \in H_0$ be in the T_0 -closure of X . Hence there is a net [7] $y_\alpha \rightarrow h_0$, $y_\alpha \in X$. If $h_0 \notin X$, and if $h_0 = \prod_{i=1}^n x_i^{\epsilon_i}$, then there is an open set $U \supset \{x_1, x_2, \dots, x_n\}$ and such that ultimately $y_\alpha \in U$. Thus we assume all $y_\alpha \in U$. Choose $f \in \mathfrak{Q}$ so that $f(x_i) = M+1$, $f(y_\alpha) = M$. Then we see $M = (M+1)^K$ where $K = \sum_{i=1}^n \epsilon_i$. For K given, we can choose M so that $M \neq (M+1)^K$ and a contradiction results.

THEOREM 4. $F(X)$ and $A(X)$ are maximally almost-periodic [6; 10; 12].

PROOF. Again we offer the proof only for $F(X)$. Let $e \neq h_0 \in H_0$. If $h_0 = \prod_{i=1}^n x_i^{\epsilon_i}$, and if $p_1, p_2, \dots, p_m$, $m \leq n$, are the distinct points among the x_i , let f be as in Lemma 2, and set $g = f/|f|$. We remark that $|q_i| = 1$, and observe that $h_0(g) \neq 1$. In view of the condition $|g| = 1$, we see that $h \rightarrow h(g)$ is a homomorphism of H_0 into the multiplicative group of unit quaternions, i.e. $[0]$ into $U(2, C)$, and $h_0(g) \neq 1$.

REMARK. Note that we have shown that the *two dimensional* unitary representations of $F(X)$ form a separating class for $F(X)$.

THEOREM 5. *Let G_A be the associated compact group (Bohr compactification) of $A(X)$ [13]. Then $\hat{G}_A$, the character group of G_A , is $K = \{f | f \in C(X), |f| \equiv 1\}$ in the discrete topology.*

PROOF. If $f \in K$, clearly f engenders a character c_f in $A(X)$ and c_f may be extended to a character of G_A . If $f_1 \neq f_2$, $c_{f_1} \neq c_{f_2}$. On the other hand, if $c \in \hat{G}_A$, c restricted to $A(X)$ provides a (possibly discontinuous) character of $A(X)$. But since $A(X)$ is generally weakened when embedded in G_A , c is in fact continuous relative to the original topology of $A(X)$. Clearly $c(x) \in K$. Finally, if $f = c(x)$, then $c_f = c$.

Analogously, we can prove Theorem 5' below. First we introduce some conventions. If $f: X \rightarrow U(n, C)$ then f engenders in a natural way a unitary representation of $F(X)$. Because $F(X)$ is maximally almost-periodic, any unitary representation of $F(X)$ may be extended uniquely to a unitary representation of the associated compact group G_F . Call f irreducible if the corresponding representation is irreducible, and call f_1, f_2 equivalent if the corresponding representations are equivalent.

THEOREM 5'. *Let f_α be a collection of functions $f_\alpha: X \rightarrow U(n_\alpha, C)$ such that (i) all f_α are irreducible; (ii) no two f_α are equivalent; (iii) any irreducible f is equivalent to one (and hence only one) f_α . Then any irreducible representation of G_F arises from an f equivalent to some f_α .*

3. Almost-periodic topology. We know that $F(X)$ has as its topology the supremum of $\mathfrak{F}$. It is of interest to investigate the infimum of $\mathfrak{F}$. To do this we introduce a special family of functions on $F_0(X)$. For each $f \in \mathfrak{Q}$, let $\tilde{f}$ denote the extension of f to $F_0(X)$ by the formula:

$$\tilde{f}\left(\prod_{i=1}^n x_i^{e_i}\right) = \prod_{i=1}^n (x_i(f))^{e_i}.$$

An elementary argument shows that $\tilde{f}$ is almost-periodic on $F_0(X)$ if and only if $|f| \equiv 1$. Let T_{AP} denote the topology on $F_0(X)$ imposed by the functions $\tilde{f}$, i.e., T_{AP} is the weakest topology with respect to which all $\tilde{f}$ are continuous.

THEOREM 6. *T_{AP} is a group topology for $F_0(X)$. In this topology, X is topologically embedded as a closed subset of $F_0(X)$. T_{AP} is the infimum of $\mathfrak{F}$.*

PROOF. The first statement may be verified by explicit calculation. For the second statement we first record the following facts and definitions.

For any quaternion q , let $e^q = \sum_{n=0}^{\infty} q^n/n!$. Then: (a) $e^q \neq 0$; (b) if r is a quaternion and $|r-1| < 1$, there is a quaternion s such that $e^s = r$; (c) for any quaternion r , $\{s | e^s = r\}$ is either empty or a discrete subset of Q , the set of all quaternions; (d) if $e^s = r$, $|r| = 1$ and if a is real then $|e^{as}| = 1$.

The infinite free set $\{q_i\}$ in Q^* may be chosen so that $|q_i| = 1$. If q is a cluster point of $\{q_i\}$, and if $1 > \epsilon > 0$ is given, let $\{r_j\}$ be a subset of $\{q_i\}$ such that $|r_j - q| < \epsilon/4$. Then $|r_j^{-1}r_j - 1| < \epsilon$, and the set $\{r_j^{-1}r_j\}$ is free. Let $e^{s_j} = r_j^{-1}r_j = t_j$. Note that $s_j \neq 0$, $|t_j| = 1$.

Since T_{AP} represents a weakening of T_0 , the embedding of X in $F_0(X)$ is continuous relative to T_{AP} . On the other hand, if W is open in X , and if $x_0 \in W$, let $f \in C(X)$ be real-valued and such that $f(x_0) = 1$, $f(X \setminus W) = 0$. Then if $g = e^{if}$ we see $|g| = 1$, whence $\tilde{g}$ is AP . Let $V = \{h | |g(h) - 1| < \epsilon = |e^i - 1|\}$. Then $x_0 V \cap X \subset W$. For, if $x \in x_0 V$, then $|g(x_0^{-1}x) - 1| < |e^i - 1|$, whence $|e^{if(x)} - e^i| < |e^i - 1|$. Thus $e^{if(x)} \neq 1$, $f(x) \neq 0$ and hence $x \in W$. Thus the embedding of X in $F_0(X)$ is open relative to T_{AP} and hence X is topologically embedded in $F_0(X)$ relative to T_{AP} .

We show next that X is closed relative to T_{AP} . Let $h_0 = \prod_{i=1}^n x_i^{\epsilon_i} \in X$, and let $p_1, p_2, \dots, p_m$ be the distinct points among $x_1, x_2, \dots, x_n$. In terms of the preliminary comments we made, choose $\epsilon > 0$ so that when $t_1, t_2, \dots, t_m$ replace $p_1, p_2, \dots, p_m$ the resulting quaternion u satisfies: $|u - 1| < 1$. Let $e^v = u$. Since $\{t_j\}$ is a free set, $u \neq 1$, $t_1, t_2, \dots, t_m$. Let $E = \{w | e^w = u\}$. Since $|u| = 1$, $u \neq 1$, we see $0, 1 \notin E$. Let L be a curve given by $\phi(\lambda)$, $0 \leq \lambda \leq 1$ and such that L avoids $E_1 = \bigcup_{j=1}^m s_j^{-1}E$, where $e^{s_j} = t_j$, $\phi(0) = 0$, $\phi(1) = 1$, and $d(L, E_1) = \Delta > 0$. Choose neighborhoods V_j of p_j such that $V_{j_1} \cap V_{j_2} = \emptyset$, $j_1 \neq j_2$, $j = 1, 2, \dots, m$, and choose $f_j \in C(X)$ so that $f_j(p_j) = 1$, $f_j(X \setminus V_j) = 0$, $0 \leq f_j(X) \leq 1$. Let $\psi_j(x) = s_j \phi(f_j(x))$, $\psi(x) = \sum_{j=1}^m \psi_j(x)$, $\gamma(x) = \exp[\psi(x)] / |\exp[\psi(x)]|$. Thus $\tilde{\gamma}$ is AP . Then $\tilde{\gamma}(h_0) = u$ and there is a $\delta > 0$ such that $|\tilde{\gamma}(x) - u| \geq \delta$ for all $x \in X$, as the following lines show: $\tilde{\gamma}(h_0) = \prod_{i=1}^n \gamma(x_i)^{\epsilon_i}$; $\gamma(x_i) = \exp[\psi(x_i)] / |\exp[\psi(x_i)]|$; $\psi(x_i) = \psi_{j_0}(p_{j_0}) = s_{j_0}$ (where $x_i = p_{j_0}$); $\exp[\psi(x_i)] = \exp[s_{j_0}] = t_{j_0}$, and $|\exp[\psi(x_i)]| = |t_{j_0}| = 1$. On the other hand, if $x \in X$, then either (a) $x \in V_{j_0}$, some j_0 or (b) $x \in (X \setminus \bigcup_{j=1}^m V_j)$. In case (a) $\tilde{\gamma}(x) = \exp[\psi_{j_0}(x)] / |\exp[\psi_{j_0}(x)]| = \exp[s_{j_0} \phi(f_{j_0}(x))] / \exp[s_{j_0} \phi(f_{j_0}(x))]$. Hence $|\exp[s_{j_0} \phi(f_{j_0}(x))]| = 1$, $\gamma(x) = \exp[s_{j_0} \phi(f_{j_0}(x))]$. The exponent is bounded away from E_1 for all j_0 and all x , whence, for some $\bar{\delta} > 0$, $|\gamma(x) - u| \geq \bar{\delta}$ in this case. In case (b) $f_j(x) = 0$, $\psi(x) = 0$, $\gamma(x) = 1$ and $\min(\bar{\delta}, |1 - u|)$ is the required δ . Hence X is closed in $F_0(X)$ relative to T_{AP} .

Finally, any group topology on $F_0(X)$ relative to which X is em-

bedded topologically is one for which all functions $\bar{f}$ are continuous. Thus T_{AP} is the infimum of $\mathfrak{F}$.

4. Identities. At the end of §1 we mentioned an interesting relationship between $F(X)$ and $A(X)$ on the one hand and Q^* and R^+ on the other. More explicitly, the situation may be described as follows: Consider the word $w = aba^{-1}b^{-1}$. In Q^* , let $N(w)$ be the subgroup generated by all quaternions arising from the substitution in w of quaternions for a and b . Then $N(w)$ is closed and normal and $Q^*/N(w) = R^+$. In $F(X)$ let $\mathfrak{N}(w)$ be the analog of $N(w)$. Then $\mathfrak{N}(w)$ is closed and normal [3] and $A(X) = F(X)/\mathfrak{N}(w)$. Furthermore, $A(X)$ arises by the constructions used when we consider R^+ -valued rather than Q^* -valued functions on X . A natural surmise might be framed as follows: Let $\{W_\lambda\}_{\lambda \in A}$ be a family of words. Let N be the closed normal subgroup generated in Q^* by substitution of quaternions for the variables in these words. Let $S = Q^*/N$. Let $\mathfrak{N}$ in $F(X)$ be the analog of N in Q^* . Then, if $B(X) = F(X)/\mathfrak{N}$, $B(X)$ arises by the constructions indicated when S -valued functions on X are employed. It is our purpose to show this conjecture is *false*.

Let $\{W_\lambda\}$ consist of one word, a^2 . It is a simple matter to verify that in this case $N = Q^*$, since every quaternion q is a square. Hence $S = 1$. On the other hand, $\mathfrak{N} \subsetneq F(X)$ as the following lines show: Let f in Ω be identically 2. Then for arbitrary $x_0 \in X$ (regarded as a subset of $F(X)$), the neighborhood U of x_0 given by $\{h \mid |h(f)x_0^{-1}(f) - 1| < 1/4\}$ does not meet $\mathfrak{N}$. Thus $\mathfrak{N}$ is a proper closed subgroup in $F(X)$ and $B = F(X)/\mathfrak{N}$ is nontrivial. But if S -valued functions are used in one construction, then, since S consists of a single number 1, the resulting construction will be the trivial group, which B is not.

5. Supplement. The author would like to thank the referee for suggesting the following generalizations of the procedures used in this paper.

Replace Q^* by a compact connected Lie group L containing a free subgroup with two generators. Hence L contains a free subgroup with infinitely many generators. Use the invariant metric in L (rather than the quaternionic norm in Q^*) to define neighborhoods as in Theorem 1. The "separating" function f of Theorem 1 is replaced by a continuous function f from X to L such that $f(x_0) = 1$, $f(X \setminus V) = a \neq 1$, and $f(X) \subset$ the geodesic arc $(1a)^\wedge$ (a segment of a one-parameter group). Specialize L to $SO(3, R)$ or $U(2, C)$ to get the results given.

Similar techniques may be applied to generalizations of Theorem 6.

The following problem now arises: Let L be the Lie-Moufang loop of all Cayley numbers of norm 1. If L can be shown to contain a free Moufang subloop of infinite rank, the procedures described lead to the construction of a free topological Moufang loop containing X .

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