A PROPERTY OF DEDEKIND'S ψ -FUNCTION

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Let n and p be used to denote positive integers and primes, respectively. Let D denote the sequence of n whose prime factors are simple and L the set of n whose prime factors are all multiple ($D \cap L = 1$). Dedekind's ψ -function has the arithmetical form

$$(1) \quad \psi(n) = \sum_{d\delta=n} dq(\delta) = n \prod_{p|n} \left(1 + \frac{1}{p}\right),$$

where $q(n)$ is the characteristic function of D .

In this note we prove that

$$(2) \quad \sum_{n=1; n \in L}^{\infty} \frac{1}{\psi(n)} = \frac{\pi^2}{6}.$$

PROOF. Let $a(n)$ denote the characteristic function of L . Since $a(n)$ and $\psi(n)$ are multiplicative, it follows formally from (1), on taking Euler products, that

$$\begin{aligned} \sum_{n=1; n \in L}^{\infty} \frac{1}{\psi(n)} &= \sum_{n=1}^{\infty} \frac{a(n)}{\psi(n)} = \prod_p \left(\sum_{i=0}^{\infty} \frac{a(p^i)}{\psi(p^i)} \right) = \prod_p \left(1 + \sum_{i=1}^{\infty} \frac{p^{-i}}{1 + p^{-1}} \right) \\ &= \prod_p \left(1 + \frac{1}{1 + p^{-1}} \cdot \frac{p^{-2}}{1 - p^{-1}} \right). \end{aligned}$$

The last expression is the product representation, $\prod_p (1 - p^{-2})^{-1}$, of the Riemann zeta-function $\zeta(s)$ when $s=2$. But $\zeta(2) = \pi^2/6$. By the convergence of the occurring products, all steps are justified and (2) is proved.

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Received by the editors May 21, 1961.