ON CERTAIN CONDITIONS FOR THE EXISTENCE OF INVARIANT LINEAR FUNCTIONALS

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1. **Introduction.** The existence of an invariant measure is a hypothesis for many theorems in topological dynamics [**2**]. N. M. Krylov and N. N. Bogoliubov have demonstrated the existence of such a measure when a one-parameter transformation group acts on a compact metric space [**1**]. Their proof involves the sequential compactness of the space of normalized measures of a compact metric space.

In this paper we shall investigate the existence of invariant positive linear functionals which is equivalent to the existence of invariant Borel measures. A different and more general approach will be used from that of the above paper. We shall not assume the space being considered to be metrizable.

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2. **Notation.** M denotes a compact space and $\mathcal{C}(M)$ the Banach space of real valued continuous functions over M together with the sup norm

$$\|\phi\| = \max\{ \,|\, \phi(x) \,|\, : x \in M\}.$$

$\mathfrak{J}(M)$ denotes the set of continuous functions mapping M into itself. α is the zero function and π the unit function of $\mathcal{C}(M)$; i.e., $\alpha(M)=0$ and $\pi(M)=1$.

DEFINITION. If $\mathcal{S}\subset\mathfrak{J}(M)$, by an $\mathcal{S}$-*functional* we mean a normalized, positive, linear functional which is invariant under $\mathcal{S}$; i.e., L is an $\mathcal{S}$-functional if L is a linear functional and

$$L(\pi) = 1, \tag{1}$$

$$\phi \geqq 0 \Rightarrow L(\phi) \geqq 0, \tag{2}$$

$$g \in \mathcal{S} \quad \text{and} \quad \phi \in \mathcal{C}(M) \Rightarrow L(\phi \circ g) = L(\phi). \tag{3}$$

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DEFINITION.

$$\mathfrak{D}(M, \mathcal{S}) = \left\{\phi = \sum_{i=1}^{n} \phi_i \circ g_i - \phi_i : \phi_i \in \mathcal{C}(M), g_i \in \mathcal{S}\right\}.$$

It is clear that $\mathfrak{D}(M, \mathcal{S})$ is a linear subspace of $\mathcal{C}(M)$.

3. **A sufficient condition.** We first prove the following:[3]

THEOREM I. *A sufficient condition for the existence of an $\mathcal{S}$-functional is that*

$$\phi \in \mathfrak{D}(M, \mathcal{S}) \Rightarrow \phi^{-1}(0) \neq \varnothing.$$

PROOF. Let $\mathcal{K}$ be the subspace spanned by $\mathfrak{D}(M, \mathcal{S})$ and π. Then for each ϕ in $\mathcal{K}$, we have $\phi = \gamma + a\pi$ where $\gamma \in \mathfrak{D}(M, \mathcal{S})$. Since $\pi^{-1}(0) = \varnothing$, $\pi \notin \mathfrak{D}(M, \mathcal{S})$, the representation is unique. Thus for every $\phi \in \mathcal{K}$ we may define

$$L'(\phi) = a.$$

It is clear that L' is a linear functional defined on $\mathcal{K}$ such that $L'(\mathfrak{D}(M, \mathcal{S})) = 0$. Moreover if $\phi = \gamma + a\pi$ then $\|\phi\| \geqq |a|$ and in case $\gamma = \alpha$, $\|\phi\| = |a|$. Therefore we have

$$\text{norm } L' = 1 = L'(\pi).$$

Now, by a well-known corollary of the Hahn-Banach theorem, we may extend L' to a linear functional, L, on $\mathcal{C}(M)$ with norm 1. L must satisfy (2), for if $\phi \in \mathcal{C}(M)$ and $\phi \geqq 0$, letting

$$c = 1/(1 + \|\phi\|),$$

we have

$$\alpha \leqq c\phi \leqq \pi,$$

and therefore

$$\|\pi - c\phi\| \leqq \|\pi\| = 1$$

so that

$$L(\pi - c\phi) \leqq 1.$$

Thus

$$L(\phi) = (1/c)L(c\phi) = (1/c)[L(\pi) - L(\pi - c\phi)] \geqq 0.$$

Finally, since $L(\mathfrak{D}(M, \mathcal{S})) = L'(\mathfrak{D}(M, \mathcal{S})) = 0$, L is an $\mathcal{S}$-functional and the theorem is proven.

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[3] This theorem may also be deduced from the more general result 2.43 of [**3**].

We next show that under certain conditions, $\mathfrak{D}(M, \mathcal{S})$ satisfies the hypothesis of Theorem I.

THEOREM II. *Let $\mathcal{S} \subset \mathfrak{J}(M)$ form a semi-group with respect to composition. Furthermore, let $x \in M$ be such that $\mathcal{S}x$ is contained in a single component of M and such that for any $g, h \in \mathcal{S}$,*

$$g(h(x)) = h(g(x)).$$

Then it follows that $\phi \in \mathfrak{D}(M, \mathcal{S}) \Rightarrow \phi^{-1}(0) \neq \varnothing$.

PROOF. Assume the contrary. Then for some $\phi = \sum_{k=1}^{n} \phi_k g_k - \phi_k$, $0 \notin \phi(M)$. Without loss of generality, we may assume

$$(') \qquad \phi(y) \geqq d > 0$$

for all $y \in M'$, the component of M containing $\mathcal{S}x$. On the other hand, we may choose an integer N sufficiently large so that

$$('') \qquad 2n\|\phi_k\| < Nd$$

for all k, $1 \leqq k \leqq n$. Now consider

$$\sum = \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} \phi(g_1^{j_1}(\cdots (g_n^{j_n}(x)) \cdots)).$$

It follows from the fact that $\mathcal{S}x \subset M'$ and $(')$ that $\sum \geqq N^n d$. On the other hand, we have

$$\sum = \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} \sum_{k=1}^{n} [\phi_k(g_k(g_1^{j_1}(\cdots (g_n^{j_n}(x)) \cdots))) - \phi_k(g_1^{j_1}(\cdots (g_n^{j_n}(x)) \cdots))]$$

$$= \sum_{k=1}^{n} \sum_{j_m=1; m\neq k}^{N} \sum_{j_k=1}^{N} [\phi_k(g_1^{j_1}(\cdots (g_k^{j_k+1}(\cdots (g_n^{j_n}(x)) \cdots)) \cdots)) - \phi_k(g_1^{j_1}(\cdots (g_n^{j_n}(x)) \cdots))]$$

since for $g, h \in \mathcal{S}$, $g(h(x)) = h(g(x))$ by hypothesis. Therefore

$$\begin{aligned} \sum &= \sum_{k=1}^{n} \sum_{j_m=1; m\neq k}^{N} [\phi_k(g_1^{j_1}(\cdots (g_k^{N+1}(\cdots (g_n^{j_n}(x)) \cdots)) \cdots)) - \phi_k(g_1^{j_1}(\cdots (g_k(\cdots (g_n^{j_n}(x)) \cdots)) \cdots))] \\ &\leqq nN^{n-1} \max_k \|\phi_k\| \\ &< N^n d, \end{aligned}$$

according to (''). This contradiction proves the theorem.

We note that in general, the condition of Theorem I is not necessary. This can readily be seen if one considers the case where M consists of a discrete space with two points and $\mathcal{S}$ is the group of permutations.

4. Necessity of the condition. If we impose certain auxiliary restrictions, the condition of Theorem I becomes a necessary one as we now show.

NOTATION. By M^x we denote the component of x in M.

THEOREM III. *If $\mathcal{S}\subset\mathfrak{I}(M)$ has the property that $\mathcal{S}x\subset M^x$, for every $x\in M$, then there exists an $\mathcal{S}$-functional only if*

$$\phi \in \mathfrak{D}(M, \mathcal{S}) \Rightarrow \phi^{-1}(0) \neq \varnothing.$$

PROOF. Let L be an $\mathcal{S}$-functional on $\mathcal{C}(M)$ and suppose there exists $\phi\in\mathfrak{D}(M, \mathcal{S})$ such that $0\notin\phi(M)$. Let

$$\begin{aligned} M' &= \{x \in M: \phi(x) > 0\}, \\ M'' &= \{x \in M: \phi(x) < 0\}. \end{aligned}$$

Since ϕ is continuous, M' and M'' are open. Since $0 \notin \phi(M)$, $M=M'\cup M''$. We also have $M'\cap M''=\varnothing$. Thus M' and M'' are closed. It follows, then, that μ', the characteristic function of M', is continuous. We may assume without loss of generality that $L(\mu')=m>0$. Furthermore, since M' is compact, we have $\phi(x)\geqq d>0$ for all $x\in M'$.

Let $\phi'\in\mathcal{C}(M)$ be defined by

$$\phi'(x) = \begin{cases} \phi(x) & \text{for } x \in M', \\ 0 & \text{for } x \in M''. \end{cases}$$

The continuity of ϕ' follows from the fact that it is continuous on the open and closed sets M' and M'' and is consistently defined on their (empty) intersection. Since $\mathcal{S}x\subset M^x$ for all x, it can easily be shown that $\phi'\in\mathfrak{D}(M, \mathcal{S})$. But from the fact that $\phi'\geqq d\mu'$ it follows that $L(\phi')\geqq dm>0$. However $L(\mathfrak{D}(M, \mathcal{S}))=0$ since L is an $\mathcal{S}$-functional. This contradiction proves the theorem.

5. Applications. The following corollaries are consequences of the above theorems.

COROLLARY. *If G is a transformation group acting on a compact space M, and either G or M is connected, then there exists a G-functional if and only if $\phi\in\mathfrak{D}(M, G)\Rightarrow\phi^{-1}(0)\neq\varnothing$.*

COROLLARY. *If G is an abelian transformation group acting on a compact space M and either G or M is connected, then there exists a G-functional.*

While it can be shown that for a single transformation, h, of a compact metric space there always exists an h-functional (by using a method similar to that of Krylov and Bogoliubov), we conclude this article by exhibiting a compact space and two homeomorphisms which do not admit a functional invariant under both of them.

EXAMPLE. Let M be the unit circle $|z| = 1$ in the complex plane. Consider the following homeomorphisms:

$$g(\exp[i\pi x]) = \begin{cases} \exp[i\pi x^2], & \text{for } 0 \leqq x \leqq 1, \\ \exp[i\pi(1 + (x-1)^{1/2})], & \text{for } 1 \leqq x \leqq 2, \end{cases}$$

$$h(\exp[i\pi(1/2 + y)]) = \begin{cases} \exp[i\pi(1/2 + y^2)], & \text{for } 0 \leqq y \leqq 1, \\ \exp[i\pi(3/2 + (y-1)^{1/2})] & \text{for } 1 \leqq y \leqq 2, \end{cases}$$

and let

$$\phi(\exp[i\pi x]) = \begin{cases} x, & \text{for } 0 \leqq x \leqq 1, \\ 2 - x, & \text{for } 1 \leqq x \leqq 2, \end{cases}$$

$$\beta(\exp[i\pi(1/2 + y)]) = \begin{cases} y, & \text{for } 0 \leqq y \leqq 1, \\ 2 - y, & \text{for } 1 \leqq y \leqq 2, \end{cases}$$

then it follows from a direct computation that $\phi(g(z)) - \phi(z) + \beta(h(z)) - \beta(z) < 0$ for all z. Thus since M is connected, Theorem III implies there cannot exist a functional invariant with respect to both g and h.

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