REFERENCES

1. E. A. Robinson, *Sums of stationary random variables*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 77-79.
2. J. L. Doob, *Stochastic processes*, New York, Wiley, 1953.

JOHN JAY HOPKINS LABORATORY FOR PURE AND APPLIED SCIENCE, GENERAL ATOMIC DIVISION OF GENERAL DYNAMICS CORPORATION, SAN DIEGO, CALIFORNIA

APPROXIMATION OF INTEGRAL CURRENTS BY CYCLES¹

HERBERT FEDERER

The purpose of this note is to prove the following theorem concerning isoperimetric ratios and cyclic approximation:

If A and B are compact Lipschitz neighborhood retracts in R^n , with $A \supset B$, and k is a positive integer then, there exists a positive number ρ with the following two properties:

- (1) For each $X \in \partial I_{k+1}(A) + I_k(B)$ there exists a $Y \in I_{k+1}(A)$ such that $\text{spt}(X - \partial Y) \subset B$ and

$$M(Y)^{k/(k+1)} \leq \rho \|X\| (A - B).$$

- (2) For each $T \in I_{k+1}(A)$ there exists an $S \in I_{k+1}(A)$ such that $\text{spt } \partial S \subset B$ and

$$M(T - S)^{k/(k+1)} \leq \rho \|\partial T\| (A - B).$$

The notation is the same as in *Normal and integral currents* by H. Federer and W. H. Fleming, Ann. of Math. vol. 72 (1960) pp. 458-520, and all references (like 6.1 below) are to that paper.

Clearly (2) follows from (1), with $X = \partial T$, $S = T - Y$.

To prove (1) let U , V , f , g , a , b , ξ , η , c_1 , c_2 , c_3 be as in 6.1 with $\Gamma = A$, let

$$\begin{aligned} \epsilon &= \inf\{a/[3n(\eta + 1)], b/(3n)\}, \\ \Phi &= R^n \cap \{x: \text{distance}(x, A) \leq 3n\epsilon\}, \\ \Psi &= R^n \cap \{x: \text{distance}(x, B) \leq 3n\epsilon\}, \end{aligned}$$

let K be the subcomplex of $\mu_e(C')$ (see 2.1, 5.1) consisting of the cubes contained in Φ , and let Ω be the group of those k dimensional chains of K which belong to

Presented to the Society, April 16, 1960; received by the editors December 2, 1960.

¹ This work was supported in part by a Sloan Fellowship.

$$\partial I_{k+1}(\Phi) + I_k(\Psi).$$

Then Ω has a finite number of free generators $\omega_1, \dots, \omega_m$ and one may choose

$$\chi_i \in I_{k+1}(\Phi) \text{ with } \text{spt}(\omega_i - \partial \chi_i) \subset \Psi$$

for $i=1, \dots, m$. Furthermore $\omega_1, \dots, \omega_m$ form a base for a vector-subspace W of $E_k(\Phi)$ normed by M , and the linear transformation

$$L: W \rightarrow N_{k+1}(\Phi)$$

such that $L(\omega_i) = \chi_i$ for $i=1, \dots, m$ has the finite norm

$$\lambda = \sup \{ N[L(w)]: w \in W, M(w) \leq 1 \}.$$

Finally let

$$\begin{aligned} \tau &= c_1^{-1} [\inf \{ b, (\eta + 2)^{-1} a \}]^k, \\ \sigma &= \sup \left\{ 2n^k \binom{n}{k}, 6n^k \binom{n}{k-1}, 4n^{k+1} \binom{n}{k} \epsilon \right\}, \\ \zeta &= \sigma + 2\sigma\lambda + (\eta + 1)3n\epsilon\eta^k(1 + 4\sigma + 2\sigma\lambda), \\ \rho &= \sup \{ c_2 \xi^k (\eta + 2)^k, \xi^k \zeta^{k/(k+1)} \tau^{-1/(k+1)} \}. \end{aligned}$$

Now suppose $X \in \partial I_{k+1}(A) + I_k(B)$.

In case $\|X\|(A-B) \leq \tau$, the conclusion of (1) follows from 6.1. Henceforth assume that $\|X\|(A-B) > \tau$, let

$$X_s = X \cap \{x: \text{distance}(x, B) > s\} \text{ for } s > 0,$$

use 3.10 with the fact that $\text{spt } \partial X \subset B$ to obtain

$$\int_0^\epsilon M(\partial X_s) ds \leq \|X\|(A-B),$$

and choose a number s for which

$$0 < s < \epsilon, \quad M(\partial X_s)\epsilon \leq \|X\|(A-B).$$

Noting that

$$\text{spt } \partial X_s \subset R^n \cap \{x: \text{distance}(x, B) = s\}$$

and applying 5.5 with $T = X_s$ one finds that

$$\begin{aligned} X_s &= P + Q + \partial S, \quad Q \in I_k(\Psi), \quad S \in I_{k+1}(\Phi), \\ P &= X + (X_s - X) - Q - \partial S \in \Omega, \\ M(P) &\leq 2\sigma\|X\|(A-B), \quad M(Q) \leq \sigma\|X\|(A-B), \\ M(S) &\leq \sigma\|X\|(A-B), \quad M(\partial S) \leq 4\sigma\|X\|(A-B). \end{aligned}$$

Accordingly

$$\begin{aligned}
 L(P) &\in I_{k+1}(\Phi), \quad \text{spt}[P - \partial L(P)] \subset \Psi, \\
 N[L(P)] &\leq \lambda M(P) \leq 2\sigma\lambda \|X\|(A - B), \\
 \Xi &= X - \partial S - \partial L(P) \\
 &= (X - X_s) + Q + [P - \partial L(P)] \in I_k(\Psi), \\
 \|\Xi\|(\Psi - B) &\leq \|X\|(A - B) + M(\partial S) + N[L(P)] \\
 &\leq (1 + 4\sigma + 2\sigma\lambda)\|X\|(A - B),
 \end{aligned}$$

with $\partial\Xi = \partial X \in I_{k-1}(B)$. Letting h be the linear homotopy from g to the identity map, and noting that

$$g(x) = x \text{ for } x \in B, \quad |g(x) - x| \leq (\eta + 1)3n\epsilon \text{ for } x \in \Psi,$$

one obtains

$$\begin{aligned}
 \Xi - g_{\sharp}(\Xi) &= \partial h_{\sharp}(I \times \Xi), \\
 M[h_{\sharp}(I \times \Xi)] &\leq (\eta + 1)3n\epsilon\eta^k \|\Xi\|(\Psi - B), \\
 X - g_{\sharp}(\Xi) &= \partial[S + L(P) + h_{\sharp}(I \times \Xi)], \\
 M[S + L(P) + h_{\sharp}(I \times \Xi)] &\leq \zeta \|X\|(A - B).
 \end{aligned}$$

Inasmuch as

$$\Phi \cup h(I \times \Psi) \subset R^n \cap \{x: \text{distance}(x, A) \leq a\},$$

one may define

$$Y = f_{\sharp}[S + L(P) + h_{\sharp}(I \times \Xi)] \in I_{k+1}(A)$$

and conclude that

$$\begin{aligned}
 X - g_{\sharp}(\Xi) &= f_{\sharp}[X - g_{\sharp}(\Xi)] = \partial Y, \\
 M(Y) &\leq \xi^{k+1}\zeta \|X\|(A - B) \\
 &\leq \xi^{k+1}\zeta r^{-1/k} [\|X\|(A - B)]^{(k+1)/k}, \\
 M(Y)^{k/(k+1)} &\leq \rho \|X\|(A - B).
 \end{aligned}$$

BROWN UNIVERSITY