A TWO-SIDED ABSORPTION PROBLEM

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1. In this note we consider a possibly asymmetric Cauchy process $X(t)$, $t \geq 0$, with transition probability densities

$$p(x, y, t) = \frac{d}{dy} \Pr\{X(t) \leq y \mid X(0) = x\}, = \frac{1}{\pi} \frac{t}{t^2 + (x - y - at)^2}.$$

Assuming as we may that the paths are right continuous, we denote by T_x the first passage time to the exterior of the open unit interval given that the process starts at x ;

$$T_x = \min\{t: |X(t)| \geq 1\}.$$

We shall obtain explicit formulas for the moments of T_x , and also for the distribution of the place of first passage $X(T_x)$. In the case of the symmetric Cauchy process these were obtained by Kac and Pollard [2] and Spitzer [4] respectively.

2. We set

$$k(x, y, t) = \frac{d}{dy} \Pr\left\{\max_{\tau \leq t} |X(\tau)| < 1 \text{ and } X(t) \leq y \mid X(0) = x\right\}.$$

Then by the extended Markov property we have

$$\begin{aligned} p(x, y, t) &= k(x, y, t) \\ (1) \quad &+ \iint_{|z| \geq 1, \tau \leq t} p(z, y, t - \tau) d_{x, \tau} \Pr\{X(T_x) \leq z, T_x \leq \tau\}. \end{aligned}$$

Set

$$K(x, y) = \int_0^\infty k(x, y, t) dt.$$

It follows from the inequalities

$$(2) \quad k(x, y, t) \leq p(x, y, t),$$

$$(3) \quad k(x, y, t) \leq \int_{-1}^1 p\left(x, z, \frac{t}{2}\right) p\left(z, y, \frac{t}{2}\right) dz,$$

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which, of course, k satisfies, that we have an estimate

$$K(x, y) \leq A \log(1 + |x - y|^{-1}).$$

If we multiply (1) by $e^{i\xi y}$ and integrate with respect to y over $(-\infty, \infty)$ and then with respect to t over $(0, \infty)$ we obtain

$$\begin{aligned} \frac{e^{i\xi x}}{|\xi| + ia\xi} &= \int_{-1}^1 e^{i\xi y} K(x, y) dy \\ &+ \frac{1}{|\xi| + ia\xi} \int_{|z| \geq 1} e^{i\xi z} d \Pr\{X(T_x) \leq z\}. \end{aligned}$$

We rewrite this as

$$(4) \quad \frac{e^{i\xi x}}{\xi} = (\operatorname{sgn} \xi + ia) \int_{-1}^1 e^{i\xi y} K(x, y) dy + \int_{|z| \geq 1} \frac{e^{i\xi z}}{\xi} d \Pr\{X(T_x) \leq z\}.$$

Now $\operatorname{sgn} \xi \int_{-1}^1 e^{i\xi y} K(x, y) dy$ is the Fourier transform of the Hilbert transform of $K(x, y)$, i.e.

$$\operatorname{sgn} \xi \int_{-1}^1 e^{i\xi y} K(x, y) dy = \frac{i}{\pi} \int_{-\infty}^{\infty} e^{i\xi u} du \int_{-1}^1 \frac{K(x, y)}{y - u} dy.$$

Therefore (4) gives easily

$$\begin{aligned} \frac{i}{2} \operatorname{sgn}(x - u) &= \frac{i}{\pi} \int_{-1}^1 \frac{K(x, y)}{y - u} dy + iaK(x, u) \\ &+ \frac{i}{2} \int_{|z| \geq 1} \operatorname{sgn}(z - u) d \Pr\{X(T_x) \leq z\}. \end{aligned}$$

Introducing the unit step function

$$\chi(y) = \begin{cases} 0, & y < 0, \\ 1, & y \geq 0, \end{cases}$$

the last equation may be written

$$(5) \quad \Pr\{X(T_x) \leq u\} - \chi(u - x) = \frac{1}{\pi} \int_{-1}^1 \frac{K(x, y)}{y - u} dy + aK(x, u).$$

This holds for all u in $|u| > 1$, and certainly almost all u in $|u| \leq 1$. In particular we obtain the singular integral equation

$$\begin{aligned} (6) \quad &\Pr\{X(T_x) \leq -1\} - \chi(u - x) \\ &= \frac{1}{\pi} \int_{-1}^1 \frac{K(x, y)}{y - u} dy + aK(x, u), \quad |u| \leq 1. \end{aligned}$$

3. It is known how to solve equations such as (6). (See, for example, [5, §4.4].) We shall reproduce the solution here since we shall need, in addition to the determination of K from the left side of (6), the determination of the left side of (5). (One can of course substitute the computed K into the right of (5), but it is even easier than that.) We state the relevant results as a lemma.

Denote by $2\pi\beta$ that argument of $(a-i)/(a+i)$ which lies between 0 and 2π and set

$$\gamma(u) = |u+1|^\beta |u-1|^{1-\beta}.$$

Then:

Assume $f \in L_2(-1, 1)$ and

$$(7) \quad \phi(u) = \frac{1}{\pi} \int_{-1}^1 \frac{f(y)}{y-u} dy + af(u) \quad p.p. \text{ in } |u| \leq 1.$$

If ϕ is defined by the right side of (7) for $|u| > 1$ (considering f to be 0 there), then

$$(8) \quad \phi(u) = -\frac{1}{\pi(a^2+1)^{1/2}\gamma(u)} \int_{-1}^1 \frac{\phi(y)\gamma(y)}{|y-u|} dy + \frac{C \operatorname{sgn} u}{\gamma(u)} |u| > 1$$

for some constant C . Moreover, with the same C ,

$$(9) \quad \begin{aligned} f(u) = & \frac{a}{a^2+1} \phi(u) - \frac{1}{\pi(a^2+1)\gamma(u)} \int_{-1}^1 \frac{\phi(y)\gamma(y)}{y-u} dy \\ & - \frac{C}{(a^2+1)^{1/2}\gamma(u)} \quad p.p. \text{ in } |u| \leq 1. \end{aligned}$$

Let us assume first that f , when extended to $|u| > 1$ by being given the value 0 there, is twice continuously differentiable. Then the function

$$F_+(w) = \frac{1}{2\pi} \int_{-1}^1 \frac{f(y)}{y-w} dy,$$

which is analytic in $\Im w > 0$, is continuous in $\Im w \geq 0$ when defined for real u by

$$(10) \quad F_+(u) = \frac{i}{2} f(u) + \frac{1}{2\pi} \int_{-1}^1 \frac{f(y)}{y-u} dy,$$

where of course the integral is now a principal value. Similarly the function

$$F_-(w) = \frac{1}{2\pi} \int_{-1}^1 \frac{f(y)}{y-w} dy$$

is analytic in $\Re w < 0$ and continuous in $\Re w \leq 0$ when defined for real u by

$$(11) \quad F_-(u) = -\frac{i}{2} f(u) + \frac{1}{2\pi} \int_{-1}^1 \frac{f(y)}{y-u} dy.$$

If we set

$$G(u) = \begin{cases} \frac{a-i}{a+i}, & |u| < 1, \\ 1, & |u| \geq 1, \end{cases} \quad \psi(u) = \begin{cases} \frac{\phi(u)}{1-ia}, & |u| < 1, \\ 0, & |u| \geq 1, \end{cases}$$

then (7), (10), and (11) imply

$$(12) \quad F_+(u) - G(u)F_-(u) = \psi(u).$$

Consider the functions $G_+(w)$ and $G_-(w)$ analytic respectively in $\Re w > 0$ and $\Re w < 0$ and given by

$$G_{\pm}(w) = (w+1)^{\beta}(w-1)^{1-\beta},$$

where the powers are determined by $0 < \arg(w \pm 1) < \pi$ in $\Re w > 0$ and $-\pi < \arg(w \pm 1) < 0$ in $\Re w < 0$. The functions $G_{\pm}$ are continuous up to the real axis and we have

$$\frac{G_-(u)}{G_+(u)} = G(u), \quad u \neq \pm 1.$$

Thus (12) can be written

$$(13) \quad G_+(u)F_+(u) - G_-(u)F_-(u) = G_+(u)\psi(u)$$

and this holds even for $u = \pm 1$. Next define the functions $\Psi_+(w)$ and $\Psi_-(w)$ analytic in $\Re w > 0$ and $\Re w < 0$ respectively and defined in their respective half-planes by

$$\Psi_{\pm}(w) = \frac{1}{2\pi i} \int_{-1}^1 \frac{G_{\pm}(y)\psi(y)}{y-w} dy.$$

Since $G_{\pm}\psi$ satisfies a Lipschitz condition, $\Psi_{\pm}$ are continuous up to the real axis and

$$\Psi_{\pm}(u) = \pm \frac{1}{2} G_{\pm}(u)\psi(u) + \frac{1}{2\pi i} \int_{-1}^1 \frac{G_{\pm}(y)\psi(y)}{y-u} dy.$$

We may now write (13) as

$$G_+(u)F_+(u) - \Psi_+(u) = G_-(u)F_-(u) - \Psi_-(u).$$

It follows that $G_+F_+ - \Psi_+$ and $G_-F_- - \Psi_-$ are analytic continuations of each other and together represent an entire function. Since this function is bounded it must be a constant $C/2$. Thus

$$F_+(u) = \frac{1}{G_+(u)} \left[\Psi_+(u) + \frac{C}{2} \right],$$

$$F_-(u) = \frac{1}{G_-(u)} \left[\Psi_-(u) + \frac{C}{2} \right].$$

We have now

$$\phi(u) = 2F_+(u), \quad |u| \geq 1,$$

$$f(u) = i[F_-(u) - F_+(u)],$$

and from these follow the identities (8) and (9).

The passage to the case of a general f is easy. Let $\{f_n\}$ be a sequence of functions, each twice continuously differentiable and vanishing outside $(-1, 1)$, which converges in L_2 to f . We obtain a corresponding sequence $\{\phi_n\}$ which, by the boundedness of the Hilbert transform in L_2 , converges in L_2 to ϕ . This shows that the identities involving ϕ_n and f_n imply the corresponding identities involving f and ϕ .

Note that our assumption $f \in L_2$ was largely a matter of convenience. In fact the identities hold assuming merely that f and ϕ both belong to L_1 , although the proof in this case makes use of rather deep properties of the Hilbert transform.

4. In our case (see (5) and (6)) ϕ is bounded. Since $\gamma(\pm 1) = 0$ it follows from (8) that

$$C = \frac{1}{\pi(a^2 + 1)^{1/2}} \int_{-1}^1 \frac{\phi(y)\gamma(y)}{\pm 1 - y} dy.$$

Since $\phi(y) = \Pr\{X(T_x) \leq -1\} - \chi(y-x)$ in $(-1, 1)$ we have two equations with which to determine C and $\Pr\{X(T_x) \leq -1\}$. We obtain

$$\Pr\{X(T_x) \leq -1\} = \frac{1}{\pi(a^2 + 1)^{1/2}} \int_x^1 \frac{\gamma(y)}{1 - y^2} dy,$$

$$C = -\frac{1}{\pi(a^2 + 1)^{1/2}} \int_x^1 \frac{y\gamma(y)}{1 - y^2} dy.$$

(Here we use the identities

$$\int_{-1}^1 \frac{\gamma(y)}{1-y^2} dy = \pi(a^2+1)^{1/2}, \quad \int_{-1}^1 \frac{y\gamma(y)}{1-y^2} dy = 0,$$

both of which follow from

$$(14) \quad \int_{-1}^1 \frac{\gamma(y)}{y-u} dy = \begin{cases} \pi(a^2+1)^{1/2} \{ \operatorname{sgn} u \gamma(u) - u \}, & |u| \geq 1, \\ \pi \{ a\gamma(u) - (a^2+1)^{1/2}u \}, & |u| \leq 1. \end{cases}$$

This in turn follows from

$$\int_{-\infty}^{\infty} \frac{G_+(y) - y}{y-u} dy = \pi i \{ G_+(u) - u \}$$

by taking imaginary parts.)

From (8), (9), (5), and (14) we have

$$(15) \quad \begin{aligned} \Pr\{X(T_x) \leq u\} &= \chi(u) + \frac{1}{\pi(a^2+1)^{1/2}} \int_x^1 \frac{\gamma(y)}{1-y^2} dy \\ &\quad - \frac{u^2-1}{\pi(a^2+1)^{1/2}\gamma(u)} \int_x^1 \frac{\gamma(y)}{(1-y^2)|y-u|} dy, \quad |u| \geq 1, \\ (16) \quad K(x, y) &= \frac{1-y^2}{\pi(a^2+1)\gamma(y)} \int_x^1 \frac{\gamma(u)}{(u-y)(1-u^2)} du \\ &\quad - \frac{a}{a^2+1} \chi(y-x), \quad |y| \leq 1. \end{aligned}$$

Nicer looking formulas may be obtained by the following method which, however, requires justification. If we differentiate both sides of (5) and use the fact that $K(x, \pm 1) = 0$ we obtain

$$\frac{d}{du} \Pr\{X(T_x) \leq u\} - \delta(u-x) = \frac{1}{\pi} \int_{-1}^1 \frac{K_v(x, y)}{y-u} dy + aK_v(x, u),$$

where δ is the Dirac delta function. Thus in equation (7) we have $\phi(u) = -\delta(u-x)$ for $|u| < 1$ and $f(y) = K_v(x, y)$. The formulas (8) and (9), if they held in this situation, would give

$$\begin{aligned} \frac{d}{du} \Pr\{X(T_x) \leq u\} &= \frac{\gamma(x)}{\pi(a^2+1)^{1/2}\gamma(u)|x-u|} + \frac{C \operatorname{sgn} u}{\gamma(u)} \\ K_v(x, y) &= -\frac{a}{a^2+1} \delta(y-x) + \frac{\gamma(x)}{\pi(a^2+1)\gamma(y)(x-y)} \\ &\quad - \frac{C}{(a^2+1)^{1/2}\gamma(y)}. \end{aligned}$$

We must have $C=0$ since $(d/du) \Pr\{X(T_x) \leq u\}$ is integrable near infinity. Thus

$$(17) \quad \frac{d}{du} \Pr\{X(T_x) \leq u\} = \frac{\gamma(x)}{\pi(a^2 + 1)^{1/2} \gamma(u) |x - u|},$$

$$(18) \quad K(x, y) = \frac{\gamma(x)}{\pi(a^2 + 1)} \int_{-1}^y \frac{du}{\gamma(u)(x - u)} - \frac{a}{a^2 + 1} \chi(y - x).$$

We leave as exercises for the reader the proofs of equivalence of (17) with (15) and (18) with (16).

5. As for the moments of T_x , we have

$$E\{T_x^n\} = n \int_0^\infty \int_{-1}^1 t^{n-1} k(x, y, t) dy dt.$$

Now

$$k(x, y, t) = \int_{-1}^1 \cdots \int_{-1}^1 k(x, z_1, \tau_1) k(z_1, z_2, \tau_2) \cdots \\ \cdots k(z_{n-2}, z_{n-1}, \tau_{n-1}) k(z_{n-1}, y, t - \tau_1 - \cdots - \tau_{n-1}) dz_1 \cdots dz_{n-1}$$

if $\tau_1, \cdots, \tau_{n-1} \geq 0$, $\sum \tau_i \leq t$. Integration of both sides over this τ region and then over $t \geq 0$ gives

$$\frac{1}{(n-1)!} \int_0^\infty t^{n-1} k(x, y, t) dt \\ = \int_{-1}^1 \cdots \int_{-1}^1 K(x, z_1) \cdots K(z_{n-1}, y) dz_1 \cdots dz_{n-1}.$$

Therefore

$$E\{T_x^n\} = n! \int_{-1}^1 \cdots \int_{-1}^1 K(x, z_1) \cdots K(z_{n-1}, y) dz_1 \cdots dz_{n-1} dy.$$

The mean first passage time has an especially simple form:

$$E\{T_x\} = \frac{\gamma(x)}{(a^2 + 1)^{1/2}}.$$

(We use (18), and a formula similar to (14) but with $1/\gamma(y)$ instead of $\gamma(y)$ in the integrand.)

6. One can obtain analogous results in case the unit interval is replaced by an arbitrary finite union of (finite or semi-infinite) intervals. The only complication is that in the analogue of the lemma of §3 several constants appear (roughly as many as the number of intervals) and are determined by solving a system of linear equations.

More serious difficulties arise in trying to extend the results to other stable processes. For general symmetric stable processes the two-sided absorption problem was solved independently by several authors [1; 3; 6]. However it seems unlikely that explicit formulas may be obtained for asymmetric stable processes with exponent $\alpha \neq 1, 2$. For $\alpha < 1$, for example, the analogue of equation (6) is

$$\begin{aligned} \Pr\{X(T_x) \leq -1\} &= \chi(u-x) \\ &= C(\alpha) \int_{-1}^1 \frac{K(x, y) \operatorname{sgn}(y-u)}{|y-u|^\alpha} dy + aK(x, u), \quad |u| \leq 1, \end{aligned}$$

where $C(\alpha)$ is a constant depending on α . From this equation it is possible to express $K(x, y)$ in terms of the eigenvalues and eigenfunctions of the kernel $|y-u|^{-\alpha} \operatorname{sgn}(y-u)$ on $(-1, 1)$, but nothing much simpler can be expected.

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