ON QUASI-REFLEXIVE BANACH SPACES

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1. Introduction. In the first part of this paper we study some properties of Schauder bases in quasi-reflexive Banach spaces. It is shown that if a quasi-reflexive space of order 1 has a Schauder basis, this basis must be either shrinking or boundedly complete. This result should be compared with the well-known theorem which states that if a reflexive Banach space has a Schauder basis, then this basis must be both shrinking and boundedly complete. If a quasi-reflexive space has a shrinking (boundedly complete) Schauder basis, then its nth conjugate space has a boundedly complete (shrinking) basis if n is odd or a shrinking (boundedly complete) basis if n is even. Furthermore if the first conjugate space of a quasi-reflexive space has a boundedly complete basis, then the space itself has a shrinking basis.

The last two sections give conditions under which a quasi-reflexive space is in fact reflexive: if every continuous linear functional attains its supremum on the unit sphere, or if the space is either smooth or rotund, then it is reflexive.

2. Definitions and notation. Let B be a Banach space, B^* and B^{**} its first and second conjugate spaces. The canonical isomorphism of B into B^{**} will be denoted by π. If A is a subspace of B we will denote by A^+ the annihilator of A in B^*. Following [1] we define B to be *quasi-reflexive of order n* if $B^{**}/\pi B$ is (finite) n-dimensional.

Let w denote the set of positive integers. A sequence $(x_i;\ i\in w)\subset B$ will be called a *basis* if for each $x\in B$ there exists a unique seqence of real numbers $(a_i;\ i\in w)$ such that $\lim_{n\in w}\|\sum_{i\leq n} a_ix_i-x\|=0$; we then write $x=\sum_{i\in w} a_ix_i$. A basis $(x_i;\ i\in w)$ for B will be called *boundedly complete* if for each sequence $(a_i;\ i\in w)$ of real numbers such that the sequence $(\|\sum_{i\leq n} a_ix_i\|;\ n\in w)$ is bounded there exists an $x\in B$ such that $x=\sum_{i\in w} a_ix_i$; it will be called *shrinking* if $\lim_{n\in w}\|x^*\|_n=0$ for each $x^*\in B^*$ where $\|x^*\|_n=\sup\{x^*(x): x=\sum_{i>n} a_ix_i \text{ and } \|x\|\leqq 1\}$, and *monotone* if for each $x=\sum_{i\in w} a_ix_i$ in B $\|\sum_{i\leq m} a_ix_i\|$ is a nondecreasing function of m.

A Banach space is *smooth* if for each $x\in B$ with $\|x\|=1$ there exists a unique $x^*\in B^*$ such that $\|x^*\|=1$ and $x^*(x)=1$. If for each $x^*\in B^*$ with $\|x^*\|=1$ there exists at most one $x\in B$ such that $\|x\|=1$ and $x^*(x)=1$, then the space is called *rotund*. It is easy to see that this terminology is equivalent to that of [2]. Finally a Banach space is

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called *subreflexive* [6] if the set of bounded linear functionals which attain their supremum on the unit sphere of B is dense in B^*.

3. **Basis in quasi-reflexive spaces.** If B is a quasi-reflexive space of order n then, by definition, we write $B^{**}=\pi B \oplus F$ where F is a finite n-dimensional subspace. It follows that if $(x_i;\ i\in w)$ is a basis for B then the sequence $(y_1^{**}, \cdots, y_n^{**}, \pi x_i;\ i\in w)$, where $y_1^{**}, \cdots, y_n^{**}$ is a basis for F, is a basis for B^{**} which we will call a basis corresponding to $(x_i;\ i\in w)$.

3.1. THEOREM. *Let B be a quasi-reflexive space. If B has a basis, then this basis is shrinking (boundedly complete) if and only if the corresponding basis in B^{**} is shrinking (boundedly complete).*

PROOF. If $(x_i;\ i\in w)$ is a shrinking basis for B and $(y_1^{**}, \cdots, y_n^{**}, \pi x_i;\ i\in w)$ a corresponding basis for B^{**}, then for $x^{***}\in B^{***}$ we have

$$\lim_m \|x^{***}\|_m = \lim_{m;\, m>n} \|x^{***}\|_m \leqq \lim_m \sup_{\|x\|\leqq 1} \left\{x^{***}(\pi x) : x = \sum_{i>m} a_i x_i\right\}.$$

By Theorem 15 [3], $B^{***}=\pi_1 B^* \oplus (\pi B)^+$, where π_1 is the canonical isomorphism of B^* into B^{***}, so if $x^{***}=\pi_1 x^* + y^{***}$, where $x^*\in B^*$ and $y^{***}\in(\pi B)^+$ we obtain

$$\lim_m \|x^{***}\|_m \leqq \lim_m \sup_{\|x\|\leqq 1} \left\{x^*(x) : x = \sum_{i>m} a_i x_i\right\} = \lim_m \|x^*\|_m = 0.$$

The corresponding basis for B^{**} is therefore shrinking.

If $(x_i;\ i\in w)$ is a boundedly complete basis for B, let $(a_i;\ i\in w)$ be a sequence of real numbers such that the sequence

$$\left(\left\| \sum_{i\leq \min(m,n)} a_i y_i^{**} + \sum_{n<i\leq m} a_i \pi x_{i-n} \right\| ;\ m\in w\right)$$

is bounded. Since B is quasi-reflexive there exists a continuous projection of B^{**} onto πB, the sequence $\{\|\sum_{n<i\leq m} a_i \pi x_{i-n}\|;\ m\in w\} = \{\|\sum_{i\leq m} a_{n+i} x_i\|;\ m\in w\}$ is bounded and there exists $x\in B$ such that $x = \sum_{i\in w} a_{n+i} x_i$. Let $y^{**} = \sum_{i\leq n} a_i y_i^{**} + \pi x$, then $y^{**} = \sum_{i\leq n} a_i y_i^{**} + \sum_{n<i} a_i \pi x_{i-n}$ and the corresponding basis in B^{**} is therefore boundedly complete. The converse in each case is clear.

3.2. COROLLARY. *Let B be a quasi-reflexive space with a monotone shrinking basis; then $B^{(n)}$ is subreflexive for all $n\geqq 0$ where $B^{(0)}=B$ and $B^{(n)}=(B^{(n-1)})^*$.*

PROOF. This is a direct consequence of [6] and the above theorem.

3.3. THEOREM. *Let B be a quasi-reflexive space. If B has a boundedly complete basis then B^* has a shrinking basis.*

PROOF. Let $(x_i;\ i\in w)$ be a boundedly complete basis for B. Let $(x_i^*;\ i\in w)$ be the sequence in B^* biorthogonal to $(x_i;\ i\in w)$ and A the closed subspace of B^* spanned by $(x_i^*;\ i\in w)$. Then by Lemma 2 [2, p. 70], $B^{**}=\pi B\oplus A^+$ and since B is quasi-reflexive A^+ must be finite n-dimensional; let $y_1^{**},\cdots,y_n^{**}$ be a basis for A^+ and pick $y_1^*,\cdots,y_n^*$ in B^* such that $y_i^{**}(y_j^*)=\delta_{ij}$. Let R be the subspace spanned by $y_1^*,\cdots,y_n^*$, then $B^*=R\oplus A$ and $(y_1^*,\cdots,y_n^*,x_i^*;i\in w)$ is a basis for B^*, since $(x_i^*;\ i\in w)$ is a basis for A.

Let $(x_i^{**};\ i\in w)$ be the sequence in B^{**} biorthogonal to $(y_1^*,\cdots,y_n^*,x_i^*,\ i\in w)$. We will show that $(x_i^{**};\ i\in w)$ spans B^{**}; by Lemma 1 [2, p. 70] this will prove that the basis for B^* is shrinking. Let $(y_1^{**},\cdots,y_n^{**},\pi x_i;\ i\in w)$ be the corresponding basis for B^{**}, then one verifies easily that $x_i^{**}=y_i^{**}$, $i\leqq n$ and $x_{n+i}^{**}=\pi x_i-\sum_{k\leqq n}y_k^*(x_i)x_k^{**}$, $i\in w$. If $x^{**}\in B^{**}$, $x^{**}=\pi x+y^{**}$ with $x\in B$ and $y^{**}=\sum_{i\leqq n}\beta_i x_i^{**}\in A^+$ and one can write $x^{**}=\sum_{i\in w}x_i^*(x)\pi x_i+\sum_{i\leqq n}\beta_i x_i^{**}=\sum_{i\in w}x_i^*(x)x_{n+i}^{**}+\sum_{i\leqq n}(\beta_i+y_i^*(x))x_i^{**}$. Therefore the sequence $(x_i^{**};\ i\in w)$ spans B^{**} and the basis for B^* is shrinking.

3.4. COROLLARY. *Let B be a quasi-reflexive space. If B has a basis which is shrinking (boundedly complete) then there exist quasi-reflexive spaces $B^{(j)}$, $j\in w$ such that $B^0=B$, $B^{(j+1)}=(B^{(j)})^*$, $B^{(2k)}$ has a shrinking (boundedly complete) basis and $B^{(2k+1)}$ has a boundedly complete (shrinking) basis.*

PROOF. The proof follows from Lemma 3.4 [1], Corollary 1 [2, p. 71] and Theorems 3.1 and 3.3 above.

Let B^* have a basis. Whether or not this implies the existence of a basis for B appears to be an open question. In case B is quasi-reflexive the following theorem provides a partial answer.

3.5. THEOREM. *Let B be a quasi-reflexive Banach space. If B^* has a boundedly complete basis then B has a shrinking basis.*

PROOF. Let $(x_i^*;\ i\in w)$ be a boundedly complete basis. By Lemma 3.4 [1] B^* is quasi-reflexive and so, as in the proof of the preceding theorem $B^{***}=\pi_1 B^*\oplus A^+$ where A is the subspace spanned by $(x_i^{**};\ i\in w)$ the sequence biorthogonal to $(x_i^*;\ i\in w)$ and A^+ is finite n-dimensional. Also $B^{**}=R\oplus A$ where R is finite n-dimensional and A is a total subspace of B^{**}. By Theorems 14 and 16 of [3] there exists an isomorphism T from B^* onto A^* such that $(Tx^*)(y^{**})=y^{**}(x^*)$ for all $x^*\in B^*$ and $y^{**}\in A$. By Theorem 3.6 of [1] there exists an

isomorphism S of A onto B. It is easy to verify that the sequence $(Sx_i^{**};\ i\in w)$ is a basis for B. To see that this basis is shrinking consider the sequence $(z_i^*;\ i\in w)$ in B^* biorthogonal to $(Sx_i^{**};\ i\in w)$. The sequence $(Tx_i^*;\ i\in w)$ in A^* is biorthogonal to $(x_i^{**};\ i\in w)$, therefore $Sy^{**}=S\sum_{i\in w}(Tx_i^*)(y^{**})x_i^{**}=\sum_{i\in w}y^{**}(x_i^*)Sx_i^{**}$ for all $y^{**}\in A$; it follows that $z_i^*(Sy^{**})=y^{**}(x_i^*)$ for all $i\in w$ and all $y^{**}\in A$; since A is total we have $T^{-1}S^*z_i^*=x_i^*$ and so $(z_i^*;\ i\in w)$ is the isomorphic image of the basis $(x_i^*;\ i\in w)$. We conclude that $(Sx_i^{**};\ i\in w)$ is a shrinking basis for B.

If B is a reflexive space and B has a basis, then it is well known (see, for example, [**2**]) that this basis must be both shrinking and boundedly complete. The following theorem holds for quasi-reflexive spaces.

3.6. THEOREM. *Let B be a quasi-reflexive space of order* 1, *then any basis for B is either shrinking or boundedly complete but not both.*

PROOF. Let $(x_i;\ i\in w)$ be a basis for the space B which is quasi-reflexive of order 1. If $(x_i;\ i\in w)$ is not shrinking, then, by Lemma 1 [**2**, p. 70], A the closed subspace spanned by $(x_i^*;\ i\in w)$, the sequence in B^* biorthogonal to $(x_i;\ i\in w)$, is a proper subspace of B^* and consequently A^+ is nonvoid. Since B has order 1 we can write $B^{**}=\pi B\oplus\{ay^{**}\}$, $y^{**}\notin\pi B$, a any real number. Let $z_0^{**}\in A^+$; then $z_0^{**}=\pi x_0+a_0y^{**}$ for some $x_0\in B$ and $a_0\neq 0$, and for any $x^*\in A$ we have $x^*(x_0)=-a_0y^{**}(x^*)$. If $z^{**}=\pi x+ay^{**}\in A^+$ we must have $x^*(x)=(a/a_0)x^*(x_0)$ for all $x^*\in A$; A being total this implies that $x=(a/a_0)x_0$ and so $z^{**}=(a/a_0)(\pi x_0+a_0y^{**})=(a/a_0)z_0^{**}$. We conclude that $A^+=\{az_0^{**}\}$ is 1-dimensional. It is easy to see that one can write $B^{**}=\pi B\oplus A^+$. By Lemma 2 [**2**, p. 70] this implies that $(x_i;\ i\in w)$ is boundedly complete. Finally if $(x_i;\ i\in w)$ were both shrinking and boundedly complete then B would be reflexive by Theorem 3 [**2**, p. 71], contradicting the hypothesis.

4. **The supremum of functionals in quasi-reflexive spaces.** James [**4**] has shown that if all bounded linear functionals on a separable Banach space attain their supremum on the unit sphere then the space is reflexive. Whether this result is true for a general Banach space is not known. However the result can be extended to non-separable quasi-reflexive Banach space.

4.1. THEOREM. *Let B be a quasi-reflexive Banach space. If all bounded linear functionals on B attain their supremum on the unit sphere of B then B is reflexive.*

PROOF. Assume the hypothesis of the theorem. If B is separable then

James' result can be applied. If B is not separable then by Theorem 4.6 of [1] there exists a nonseparable reflexive subspace Z such that B/Z is separable.

The mapping T of Z^+ onto $(B/Z)^*$ defined by $Tx^*(x+Z)=x^*(x)$ for all $x^*\in Z^+$ and $x\in B$, is an isometric isomorphism. If $x^*\in Z^+$ there exists $x_0\in B$ such that $\|x_0\|=1$ and $x^*(x_0)=\|x^*\|$, and therefore $Tx^*(x_0+Z)=x^*(x_0)=\|x^*\|=\|Tx^*\|$. Every functional on B/Z attains its supremum on the unit sphere of B/Z consequently B/Z, as well as Z is reflexive. It follows that B is reflexive.

5. Rotundity and smoothness in quasi-reflexive spaces.

5.1. THEOREM. *If B is a quasi-reflexive Banach space, there exists an equivalent norm for B such that if B is either rotund or smooth in this norm, then B is reflexive.*

PROOF. By Theorem 3.5 [1] there exists an equivalent norm for B and quasi-reflexive Banach spaces $B^{(-5)}$, $B^{(-4)}$ and $B^{(-1)}$ such that $(B^{(-5)})^{****}=B^{(-1)}$, $(B^{(-4)})^{****}=B$ and $(B^{(-1)})^*=B$. By Theorem 20 [3], if B is rotund in this norm we must have $B^{(-4)}$ reflexive. It follows that B is reflexive. If B is smooth, then by [5, p. 12] $B^{(-1)}$ is rotund and so $B^{(-5)}$ is reflexive; B is therefore reflexive.

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