THE TRACE IN FINITE OPERATOR ALGEBRAS

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1. **Introduction.** In [3] we gave a short proof of the additivity of the trace in a factor of type II_1 . In the present note, we supply the comments necessary to extend this proof to von Neumann algebras of type II_1 . J. Dixmier proves this in [2, Chapter III, §8] with the aid of his "Théorème d'approximation" (a result of independent interest and importance in the subject; cf. [2, Chapter III, §5 and §8]). He uses it in passing from his approximate trace which is local to a global trace. For a factor, "local" is the same as "global"; and the passage from an approximate trace to a global trace can be easily effected in [3] by reference to the w^* -compactness of the state space. In §2, we define a "center state." The systematic use of this concept yields a global approximate trace. An easy extension of Alaoglu's result [1] establishes the compactness of the center states in an appropriate topology. We have put this extension in a general form which seems to unify the treatment of the various standard variants of Alaoglu's result. In §3, we list those modifications of the proof in [3] which make it valid for von Neumann algebras of type II_1 .

2. **A weak-compactness principle.** Let V and V' be real or complex linear spaces and F a separating family of linear functionals on V' . Denote by t' the weak- F topology on V' ; and let S be a subset of V which spans it linearly, S' a t' -compact subset of V' , $\mathcal{S}$ the set of linear transformations of V into V' which map S into S' , and t the point-open topology on the space of linear transformations of V into V' , where V' is taken in the t' topology (i.e. a subbase for the open sets of t is the family of sets consisting of all linear transformations which map a given point in V into a given t' -open set in V').

THEOREM. $\mathcal{S}$ is t -compact.

PROOF. Each T in $\mathcal{S}$ corresponds to a point T' in X , the product of copies of S' indexed by points of S , defined by $T'(x) = Tx$, for each x in S . The mapping, $T \rightarrow T'$, is one-one and a homeomorphism of $\mathcal{S}$ in the t -topology with its image, $\mathcal{S}'$, in the topology induced by the product topology on X (each copy of S' taken in the t' topology).

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Since X is compact in this topology, it remains to show that $\mathcal{S}'$ is closed in X .

Let p be a limit point of $\mathcal{S}'$ in X , and let $p(x)$ be its coordinate in S' corresponding to x in S . Then p is a mapping of S into S' . Let R be its (possibly many-valued) linear extension to V (recall that S spans V). It suffices to prove that $\alpha_1 p(x_1) + \cdots + \alpha_n p(x_n) = 0$ if $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$, with $x_1, \cdots, x_n$ in S , in order to show that R is single-valued and linear. Let f in F and $\epsilon > 0$ be given. With $c = 1 + |\alpha_1| + \cdots + |\alpha_n|$, choose T in $\mathcal{S}$ such that $|f[T'(x_j) - p(x_j)]| < \epsilon/c$. Then

$$\begin{aligned} & |f[\alpha_1 p(x_1) + \cdots + \alpha_n p(x_n)]| \\ &= |f[\alpha_1 T'(x_1) + \cdots + \alpha_n T'(x_n) - \alpha_1 p(x_1) - \cdots - \alpha_n p(x_n)]| < \epsilon. \end{aligned}$$

Since F is separating for V' and ϵ is an arbitrary positive number, $\alpha_1 p(x_1) + \cdots + \alpha_n p(x_n) = 0$. Clearly, $p = R' \in \mathcal{S}'$.

COROLLARY. *Let (S_α) be a family of subsets of S , (S'_α) a family of t' -compact subsets of S' , and $\mathcal{S}_0$ the subset of $\mathcal{S}$ consisting of those transformations which map S_α into S'_α , for each α . Then $\mathcal{S}_0$ is t -compact.*

PROOF. Let $\mathcal{S}_\alpha$ be the subset of $\mathcal{S}$ consisting of those transformations which map S_α into S'_α . Then $\mathcal{S}_0 = \bigcap_\alpha \mathcal{S}_\alpha$. Let $\mathcal{S}_y$ be the subset of $\mathcal{S}$ consisting of those transformations which map a given point, y , in S_α into S'_α . Then $\mathcal{S}_\alpha = \bigcap_y \mathcal{S}_y$. Finally, each $\mathcal{S}_y$ is t -closed, by definition of the point-open topology.

Some special instances follow:

1. V and V' are normed spaces, with V' reflexive, S and S' are their unit balls, and F is the dual of V' .

2. With V' , the scalars and the rest as in 1 gives Alaoglu's result.

3. With V and V' the same Hilbert space and the rest as in 1, yields the compactness of the unit ball in all bounded operators provided with the weak-operator topology, and from this, the compactness of the unit ball in a von Neumann algebra.

4. From 3, with V' a von Neumann algebra, S' its unit ball, F all functionals of the form $T \rightarrow (Tx, y)$, with x and y vectors in the underlying Hilbert space—in particular, with V a von Neumann algebra and S its unit ball—we have the compactness of the unit ball in the space of continuous linear mappings of V into V' . We call the t topology on this space of mappings the weak-operator topology.

5. With V , V' , and F , as in 4, S , S' the positive operators in the unit spheres of V and V' , and $S_1 = S'_1 = (I)$, we have the weak-operator compactness of the set of positive normalized linear mappings of

V into V' . With V' the scalars, we have the compactness of the state space.

6. Our application will be to the situation in 5, where V' is the center of V , $S_\alpha = S'_\alpha$ ranging through the single-point sets of the center. The resulting positive linear mappings into the center which are the identity on the center will be called "center states."

If r is a center state of $\mathfrak{A}$ with center $\mathfrak{C}$, then $r(CT) = Cr(T)$, for each C in $\mathfrak{C}$ and T in $\mathfrak{A}$, as follows from the first statement of [5, Lemma 2]. However, both follow from:

LEMMA. *If r is a positive linear mapping of one C^* algebra, $\mathfrak{A}$, into another, $\mathfrak{B}$, which maps an abelian self-adjoint subalgebra, $\mathfrak{A}$, of $\mathfrak{A}$ homomorphically into the center of $\mathfrak{B}$ and I onto I ; then $r(CA) = r(C)r(A)$, for each C in $\mathfrak{A}$ and each A in $\mathfrak{A}$.*

PROOF. With p a pure state of $\mathfrak{B}$, p is multiplicative on the center of $\mathfrak{B}$, so that $pr([C - pr(C)I]^2) = 0$, for each C in $\mathfrak{A}$; and $C - pr(C)I$ is in the left kernel of pr . Thus, $0 = pr[CA - pr(C)A] = pr(CA) - pr(C)pr(A)$. With $\mathfrak{B}$ the scalars, r is a state of $\mathfrak{A}$ which is multiplicative (i.e. pure) on $\mathfrak{A}$; and, from the above, $r(CA) = r(C)r(A)$. Applying this to p on $\mathfrak{B}$, we have $pr(C)pr(A) = p[r(C)r(A)]$; whence $r(CA) = r(C)r(A)$, in the general case.

In the above proof, we need only that r maps $\mathfrak{A}$ into an abelian self-adjoint subalgebra of $\mathfrak{B}$ with the property that the set of state extensions of its pure states separate $\mathfrak{B}$. However:

COROLLARY. *The set of states of $\mathfrak{A}$ which restrict to pure states of the abelian self-adjoint subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ separates $\mathfrak{A}$ if and only if $\mathfrak{A}_0$ is contained in the center of $\mathfrak{A}$.*

PROOF. Since each pure state of $\mathfrak{A}$ is multiplicative on the center, it is pure on each subalgebra of the center. If p restricts to a pure state of $\mathfrak{A}_0$, then, from the above lemma, $p(AC - CA) = 0$, for each A in $\mathfrak{A}$ and C in $\mathfrak{A}_0$. By hypothesis, then, $AC = CA$.

3. The trace. Modify the proof of [3, §2] as follows, to get one valid for von Neumann algebras of type II_1 . Read "center state" for "state," "positive multiple of a center state" for "positive linear functional," "weak-operator compactness" for "weak compactness," "projection with scalar dimension" for "nonzero projection," "von Neumann algebra of type II_1 " for "factor of type II_1 ." As dimension function, use the one constructed in [4, §3.2]. To construct η (i.e., a normal center state), in the last paragraph of [3], let R be an abelian projection with central carrier I in the commutant, $\mathfrak{C}'$, of the center,

$\mathfrak{C}$, of M . Then $R\mathfrak{C}'R = \mathfrak{C}R$, and the representation CR of an operator in $\mathfrak{C}R$ is unique, since R has central carrier I . Let $\eta(T) = C$, where $RTR = CR$, for T in M .

ADDED IN PROOF. The last paragraph of [3] should be replaced by the following for the present proof. All projections mentioned lie in M and are nonzero. If each subprojection of G contains a subprojection F such that $\eta(F)D(G) < D(F)\eta(G)$, then a maximal orthogonal family $\{G_\alpha\}$ of subprojections of G such that $\eta(G_\alpha)D(G) < D(G_\alpha)\eta(G)$ has sum G . Since η is normal, $\sum_\alpha \eta(G_\alpha)D(G) = \eta(\sum_\alpha G_\alpha)D(G) = \eta(G)D(G) < \sum_\alpha D(G_\alpha)\eta(G) = D(G)\eta(G)$. Thus G has a subprojection F each subprojection F' of which satisfies $\eta(F')D(G) < D(F')\eta(G)$. If $D(F')\eta(G) \not\leq \eta(F')D(G)$, then for some central projection Q , $Q\eta(F')D(G) = \eta(F'Q)D(G) < D(F'Q)\eta(G)$. But $F'Q$ is a subprojection of F contradicting the property of F . Thus $D(F')\eta(G) \leq \eta(F')D(G)$ for each $F' \leq F$. The same argument with each inequality sign reversed shows that G has a subprojection H such that each $H' \leq H$ satisfies $D(H')\eta(G) \geq \eta(H')D(G)$. Take I for G , so that $D(F') \leq \eta(F')$ when $F' \leq F$. Let $a = \inf \{ \sup \{ b : bD(F') \leq \eta(F') \} : F' \leq F \}$. Then $a \geq 1$, by choice of F , and $aD(E') \leq \eta(E')$, for each $E' \leq F$. Choose $F' \leq F$, such that $cD(F') \not\leq \eta(F')$, where $c = (n+1)a/n$. For a suitably chosen central projection Q , $\eta(F'Q) < cD(F'Q)$. We may assume that $\eta(F') < cD(F')$. Next, choose $E \leq F'$ such that $D(E')\eta(F') \geq \eta(E')D(F')$, when $E' \leq E$. Then $\eta(E')D(F') \leq cD(E')D(F')$. Since $E' \leq F' \leq F$, $aD(E') \leq \eta(E')$, and $\eta(E') \leq \eta(F') < cD(F')$. We may, therefore, write $\eta(E') \leq cD(E')$ in place of $\eta(E')D(F') \leq cD(E')D(F')$. Thus $aD(E') \leq \eta(E') \leq cD(E')$.

Using η/a for ρ , from the first part of the proof in [3], there is a center state ψ_n of MQ such that $\psi_n(B^*B) \leq (n+1)\psi_n(BB^*)/n$, for each B in MQ , where Q is a central projection such that $D(E) \geq Q/m$, m an integer. The preceding paragraph proves that a maximal orthogonal family of central projections $\{Q_\alpha\}$ such that there exist center states $\{\phi_n\}$ of MQ_α satisfying $\phi_n(B^*B) \leq (n+1)\phi_n(BB^*)/n$, for each B in MQ_α , has sum I . Then ϕ_n defined on M by: $\phi_n(B) = \sum_\alpha \phi_n(BQ_\alpha)$, is a center state of M such that $\phi_n(B^*B) \leq (n+1)\phi_n(BB^*)/n$, and the proof is complete.

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