STARLIKE HYPERGEOMETRIC FUNCTIONS

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Let a, b, c be complex numbers with $c \neq 0, -1, -2, \dots$. For the hypergeometric function $F(a, b, c; z)$, which is defined in $|z| < 1$ by the series

$$(1) \quad 1 + \frac{ab}{c1!} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \dots,$$

the identities

$$(2) \quad cF(a, b+1, c; z) = aF(a+1, b+1, c+1; z) + (c-a)F(a, b+1, c+1; z),$$

$$(3) \quad cF'(a, b, c; z) = abF(a+1, b+1, c+1; z)$$

are readily established.

When in the identity

$$c(1-z)F(a, b+1, c; z) = cF(a, b, c; z) - (c-a)zF(a, b+1, c+1; z),$$

which is due to Gauss [1, p. 130], $cF(a, b+1, c; z)$ is replaced by the right member of (2) the identity

$$\begin{aligned} a(1-z)F(a+1, b+1, c+1; z) \\ = cF(a, b, c; z) - (c-a)F(a, b+1, c+1; z) \end{aligned}$$

is obtained. By (3) this can be rewritten for $z \neq 1$ as

$$(4) \quad z \frac{F'(a, b, c; z)}{F(a, b, c; z)} = \frac{bz}{1-z} \left[1 - \frac{c-a}{c} \frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} \right],$$

which is instrumental in the proof of the following:

THEOREM. *Let $0 < a < c$ and $-1 < b \leq a$. Then the function $u(z) = zF(a, b, c; z)$ is univalent and starlike with respect to the origin for $|z| < \rho$, where*

$$(i) \quad \rho = \frac{1}{1-b} \quad \text{when} \quad -1 < b < 0;$$

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$$(ii) \quad \rho = 1 \quad \text{when} \quad 0 \leq b \leq 2;$$

$$(iii) \quad \rho = \frac{1}{b-1} \quad \text{when} \quad b > 2.$$

PROOF. A sufficient condition for $u(z)$ to be univalent and starlike in $|z| < r$ is [4]

$$(5) \quad \operatorname{Re} s(z) \geq 0, \quad |z| < r,$$

where

$$(6) \quad s(z) = z \frac{u'(z)}{u(z)} = z \frac{F'(a, b, c; z)}{F(a, b, c; z)} + 1.$$

A result of Paydon and Wall [2] (see also [5, p. 46]) shows that if $0 < g_n < 1$ ($n = 1, 2, \dots$), then the function $f(z)$ defined in $|z| < 1$ by the continued fraction

$$(7) \quad f(z) = \frac{g_1}{1 - \frac{(1 - g_1)g_2z}{1 - \frac{(1 - g_2)g_3z}{1 - \dots}}}$$

satisfies

$$(8) \quad \left| f(z) - \frac{1}{2 - g_1} \right| \leq \frac{1 - g_1}{2 - g_1}, \quad |z| < 1.$$

A result of Wall [5, p. 283] states that when $f(z)$ is given by (7),

$$(9) \quad \frac{1 - f(z)}{1 - zf(z)} = \frac{1 - g_1}{1 - \frac{g_1(1 - g_2)z}{1 - \frac{g_2(1 - g_3)z}{1 - \dots}}},$$

where the right member of (9) is obtained from the right member of (7) by replacing g_n by $1 - g_n$. In this case inequality (8) becomes

$$\left| \frac{1 - f(z)}{1 - zf(z)} - \frac{1}{1 + g_1} \right| \leq \frac{g_1}{1 + g_1}, \quad |z| < 1,$$

which is equivalent to the inequality

$$(10) \quad \left| f(z) - \frac{g_1(1 - \bar{z})}{1 + g_1 - z - \bar{z} + (1 - g_1)z\bar{z}} \right| \leq \frac{g_1 |1 - z|}{1 + g_1 - z - \bar{z} + (1 - g_1)z\bar{z}}, \quad |z| < 1.$$

For

$$g_{2n+1} = \frac{c-a+n}{c+2n}, \quad g_{2n+2} = \frac{c-b+n}{c+2n+1}, \quad (n = 0, 1, 2, \dots),$$

the function $(c-a)F(a, b+1, c+1; z)/cF(a, b, c; z)$ is identified with the function $f(z)$ of (7) [5, p. 339] and

$$s(z) = 1 + \frac{bz}{1-z} [1 - f(z)].$$

The requirement $0 < g_n < 1$ is equivalent to the conditions $0 < a < c$, $-1 < b < c$, and the second condition can be replaced by $-1 < b \leq a$ since $F(a, b, c; z)$ is symmetric in a and b . Thus, under the hypotheses of the Theorem, it follows from (10) that

$$|s(z) - C| \leq R, \quad |z| < 1,$$

where

$$C = 1 + \frac{bz(c - a\bar{z})}{2c - a + c(z + \bar{z}) + a\bar{z}\bar{z}},$$

$$R = \frac{|bz|(c - a)}{2c - a + c(z + \bar{z}) + a\bar{z}\bar{z}},$$

and hence (5) will hold if the function

$$H(\theta) = \operatorname{Re} C - R$$

is non-negative for $z = re^{i\theta}$, $r < 1$.

When $-1 < b < 0$,

$$H(\theta) = 1 + \frac{br(c \cos \theta - ar + c - a)}{2c - a - 2cr \cos \theta + ar^2},$$

and a simple computation shows that the minimum of $H(\theta)$ occurs for $\theta = 0$. The statement (i) is then easily obtained from the requirement

$$H(0) = 1 + \frac{br}{1-r} \geq 0.$$

Similarly, when $b > 0$, the minimum of

$$H(\theta) = 1 + \frac{br(c \cos \theta - ar - c + a)}{2c - a - 2cr \cos \theta + ar^2}$$

occurs when $\theta = \pi$ and the requirement

$$H(0) = 1 - \frac{br}{1+r} \geq 0$$

leads to the statements (ii) and (iii).

It is evident that the result (ii) is sharp since $u(z)$ is not regular at $z=1$ except in the case where $b=0$.

A necessary and sufficient condition for a nonconstant regular function $F(z)$ to be univalent in $|z| < \rho$ and to map $|z| < \rho$ onto a convex region is that $zF'(z)$ be univalent and starlike with respect to the origin for $|z| < \rho$ [3, p. 221]. Since

$$czF'(a, b, c; z) = abzF(a+1, b+1, c+1; z)$$

by (3), application of the Theorem yields the following:

COROLLARY. *The function $F(a, b, c; z)$, where $-1 < a < c$, $-2 < b \leq a$, $a \neq 0$, $b \neq 0$, $c \neq 0$, is univalent in $|z| < \rho$ and maps $|z| < \rho$ onto a convex region if*

- (i) $\rho = -1/b$ when $-2 < b < -1$,
- (ii) $\rho = 1$ when $-1 \leq b \leq 1$, $b \neq 0$,
- (iii) $\rho = 1/b$ when $b > 1$.

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