A QUESTION OF KATĚTOV CONCERNING THE HILBERT PARALLELOTOPE

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For a linear subspace L of a linear space E , it is customary to define the *dimensional deficiency* of L in E as the algebraic dimension of a linear subspace which is supplementary to L in E . The notion is useful even when E is infinite-dimensional. At a recent conference in Warsaw, Professor Katětov asked whether one can define an analogous notion in purely topological terms, at least for the closed subsets of a compact metric space. Such a notion would be useful in applying dimension theory to infinite-dimensional compacta. He suggested the following test-problem: *Does the Hilbert parallelopete* $P = \{x = (x^1, x^2, \dots) \in l^2: \text{always } x_n \in [0, 1/n]\}$ *admit a self-homeomorphism which carries* A *onto* B , *where* $A = \{x \in P: x^1 = 0\}$ *and* $B = \{x \in P: x^1 = x^2 = 0\}$? Since the sets A and B are intuitively of different dimensional deficiency in P , the existence of such a homeomorphism would suggest that the topological notion mentioned above cannot be formulated in a useful way. In the present note, such a homeomorphism is constructed by means of some mapping techniques originated by Keller [1] and developed further by the author [2; 3; 4]. The following more general theorem is proved:

THEOREM. *Suppose that (for $\gamma = a, b$) K_γ is a compact convex subset of a normed linear space, H_γ is a proper supporting hyperplane² of K_γ , and C_γ is an infinite-dimensional closed convex subset of $K_\gamma \cap H_\gamma$. Then there is a homeomorphism of K_a onto K_b which carries C_a onto C_b .*

The theorem expresses, in particular, a homogeneity property of infinite-dimensional compact convex sets. Other homogeneity properties of such sets have been established by Keller [1] and the author [2; 3; 4].

A convex subset S of a normed linear space is said to be *elliptically convex* provided whenever x and y are distinct points of S , the open segment $]x, y[$ consists entirely of nonsupport points of S . Let Z denote the Hilbert space $L^2]0, \infty[\times R$, where R is the real number space and $\|(f, r)\| = (\|f\|^2 + r^2)^{1/2}$. For each $r \in R$, let X_r denote the

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² That is, K_γ is not contained in H_γ but is contained in one of the closed halfspaces bounded by H_γ .

hyperplane $L^2]0, \infty[\times\{r\}\subset Z$. Let C' be an (easily constructed) compact subset of X_0 which is elliptically convex relative to X_0 , and let K' denote the convex hull of C' together with the point $(0, 1)$. We shall prove the theorem by showing that if K_γ and C_γ are as described, then there is a homeomorphism ζ_γ of K_γ onto K' which carries C_γ onto C' . Then the transformation $\zeta_\gamma^{-1}\zeta_\alpha$ will be the desired homeomorphism of K_α onto K_β . Since K_α and K_β enter symmetrically in the hypotheses, we may restrict our attention to a single set K , hyperplane H , and subset C satisfying the conditions of the theorem. In constructing a homeomorphism ζ of K onto K' such that $\zeta C = C'$, we shall rely on the following facts:

I. (See p. 38 of [3].) In a separable normed linear space, a complete convex set which does not lie in any hyperplane must admit a non-support point.

II. (See the proofs of IV 1.2 in [2] and 5.4 in [3].) Suppose B is a bounded closed convex subset of a normed linear space E , $0 \in \text{int } B$, and μ is the gauge functional of B . Suppose S is a convex subset of E with $0 \in S$, and $Tx = (1 + \mu x)^{-1}x$ for each $x \in S$. Then T is a homeomorphism and TS is convex. Suppose further that 0 is a non-support point of S , p_1 and p_2 are distinct points of TS , and $q \in]p_1, p_2[$. If q is a support point of TS , then the entire segment $[p_1/\mu p_1, p_2/\mu p_2]$ lies in the boundary of B .

III. (See the proof of III 1.1 of [2].) With Z and X_0 as above, let $Y = L^2]0, \infty[\times[0, 1] \subset Z$ and let W denote the subspace of X_0 consisting of all points $(f, 0)$ for which $f \in L^2]0, \infty[$ and $f = 0$ almost everywhere on $[0, 1]$. For each $f \in L^2]0, \infty[$ and $t \in [0, 1]$, let the function $f_t \in L^2]0, \infty[$ be defined as follows:

$$f_t x = \begin{cases} f(x + t - 1) & \text{when } x \in [1, \infty], \\ tf(x) & \text{when } x \in]0, 1]. \end{cases}$$

For each $(f, r) \in Y \cup X_0$, let $h(f, r) = (f_r, r)$. Then h is a homeomorphism of $Y \cup X_0$ onto $Y \cup W$, each restriction $h|_{X_r}$ is an affine transformation, $hX_r = X_r$ for $r \in]0, 1]$, and $hX_0 = W$.

Now we are faced with a compact convex subset K of a normed linear space, a proper supporting hyperplane H of K , and an infinite-dimensional closed convex subset C of $K \cap H$. We wish to construct a homeomorphism ζ of K onto the set $K' \subset Z$ such that $\zeta C = C'$. The homeomorphism ζ will be obtained as the composition of homeomorphisms $\zeta_1, \dots, \zeta_7$ (i.e., $\zeta = \zeta_7 \zeta_6 \zeta_5 \zeta_4 \zeta_3 \zeta_2 \zeta_1$). We set $K_0 = K$, $C_0 = C$, and agree that whenever a homeomorphism ζ_i of K_{i-1} has been defined, then $K_i = \zeta_i K_{i-1}$ and $C_i = \zeta_i C_{i-1}$. Let us now define the homeomorphisms ζ_i .

(ζ_1) The affine hull $\text{aff } K_0$ of K_0 is separable and thus admits a total sequence f_α of continuous affine functionals which are uniformly bounded on K_0 and such that $H \cap \text{aff } K_0 = \{p: f_1 p = -1\}$ and $H' \cap \text{aff } K_0 = \{p: f_1 p = 1\}$, where H' is K_0 's other supporting hyperplane which is parallel to H . For each $p \in K_0$, let $\zeta_1 p = (f_1 p, f_2 p/2, \dots, f_n p/n, \dots) \in l^2$. Then ζ_1 is an affine homeomorphism of K_0 onto a compact convex subset K_1 of l^2 , the set $C_1 = \zeta_1 C_0$ lies in the intersection of K_1 with its supporting hyperplane $J_{-1} = \{x \in l^2: x^1 = -1\}$, and K_1 is supported also by the hyperplane $J_1 = \{x \in l^2: x^1 = 1\}$.

(ζ_2) Let $v_{-1} \in C_1 \subset K_1 \cap J_{-1}$ and $v_1 \in K_1 \cap J_1$. For each $x \in l^2$ let $\pi x = x - x^1 \delta_1 = (0, x^2, x^3, \dots)$. Then define ζ_2 as follows:

$$(\zeta_2 x)^1 = x^1;$$

$$\pi(\zeta_2 x) = \pi x + \frac{1}{2} x^1 (\pi v_{-1} - \pi v_1) - \frac{1}{2} (\pi v_1 + \pi v_{-1}).$$

Then ζ_2 is an affine homeomorphism, $\zeta_2 v_{-1} = -\delta_1$, and $\zeta_2 v_1 = \delta_1$. The set K_2 is supported by the hyperplanes J_{-1} and J_1 , $-\delta_1 \in C_2 \subset K_2 \cap J_{-1}$, and $\delta_1 \in K_2 \cap J_1$.

(ζ_3) Let B' be a spherical cell in l^2 which is tangent to the hyperplane J_1 at the point δ_1 and such that $C_2 \subset B'$. Let B denote the set of all $x \in B'$ for which $x^1 + 1$ is no less than the distance from πx to the set πC_2 . Then $B \cap J_{-1} = C_2$, $B \cap J_1 = \{\delta_1\}$, and B is a bounded closed convex set. Let μ denote the gauge functional of B , and for each $x \in K_2$ let $\zeta_3 x = (1 + \mu x)^{-1} x$. Since $0 \in K_2$, it follows from II that ζ_3 is a homeomorphism and the set K_3 is convex. Clearly K_3 is supported by the hyperplanes $J_{-1/2}$ and $J_{1/2}$; $K_3 \cap J_{-1/2} = C_3$ and $K_3 \cap J_{1/2} = \{\delta_1/2\}$.

(ζ_4) By well-known properties of Hilbert spaces, there exists an affine isometry ζ_4 of l^2 onto Z such that $\zeta_4(\delta_1/2) = (0, 1)$ and $\zeta_4 J_{-1/2} = X_0$. Then of course $K_4 \subset Y \cup X_0$ and $C_4 \subset X_0$. From I it follows that the set $K_4 \cap X_{1/2}$ admits a nonsupport point z relative to $X_{1/2}$ and the set C_4 admits a nonsupport point z_0 relative to the flat $V = \text{cl aff } C_4$, the smallest closed flat containing C_4 . For $r \in [0, 1]$, let z_r be defined as follows:

$$z_r = \begin{cases} (1 - 2r)z_0 + 2rz & \text{for } r \in [0, 1/2] \\ (2 - 2r)z + (2r - 1)(0, 1) & \text{for } r \in [1/2, 1]. \end{cases}$$

Then for each $r \in]0, 1[$, z_r is a nonsupport point of the set $K_4 \cap X_r$ relative to X_4 .

(ζ_5) Let h be as in III. Then $\text{cl } hV$ is a flat of infinite dimension and infinite deficiency in X_0 , so (again by well-known properties of

Hilbert spaces) there is a linear isometry η of Z onto Z which carries Y onto Y , X_0 onto X_0 , and $\text{cl } hV$ onto W . For each $x \in K_4$, let $\zeta_5 x = h^{-1}\eta h x$. Then ζ_5 is a homeomorphism. Although the set $K_5 = \zeta_5 K_4$ may fail to be convex, the following assertions can be verified:

- (i) for each $r \in [0, 1]$, the set $D_r = K_5 \cap X_r$ is compact and convex;
 - (ii) $D_0 = C_5$ and $D_1 = \{(0, 1)\}$;
 - (iii) for each $r \in [0, 1[$, $\text{cl aff } D_r = X_r$;
 - (iv) the function $D_r | r \in [0, 1]$ is continuous, where $\{D_r : r \in [0, 1]\}$ is metrized by means of the Hausdorff metric;
 - (v) with $q_r = \zeta_5 z_r$, q_r is a nonsupport point of the set D_r (relative to X_r) for each $r \in [0, 1[$;
 - (vi) the function $q_r | r \in [0, 1]$ is continuous.
- (ζ_6) For each $(x, r) \in K_5$ let

$$\zeta_6(x, r) = (q_r + (1 + \|x - q_r\|)^{-1}(x - q_r), r).$$

From II and (v) above it follows that for each $r \in [0, 1[$, the set $G_r = \zeta_6 D_r$ is elliptically convex relative to X_r . And it is evident that conditions (i)–(vi) above remain satisfied when D_r is replaced by G_r in (i)–(v), K_5 by K_6 in (i), and C_5 by C_6 in (ii).

(ζ_7) Now recall the definition of the set K' , and for each $r \in [0, 1]$, let $K'_r = K' \cap X_r$. Then $K'_1 = \{(0, 1)\} = G_1$, and for each $r \in [0, 1[$ the sets G_r and K'_r are both compact elliptically convex subsets of X_r . Let $\zeta_7(0, 1) = (0, 1)$, and for each $r \in [0, 1[$ let ζ_7 be the Keller-homeomorphism of G_r onto K'_r (as discussed in [1; 2; 3; 4]). Then clearly $\zeta_7 K_6 = K'$ and $\zeta_7 C_6 = C'$. That ζ_7 is a homeomorphism follows from condition (v) above in conjunction with the continuity property of the Keller-homeomorphisms expressed in 4.2 of [3]. The proof is complete.

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