CLASSES OF p-VALENT STARLIKE FUNCTIONS

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1. Introduction. The winding number associated with a starlike function exhibits a certain monotonicity property (Theorem 1 below). This property is used to show that several alternatives to the definition of the class $S(p)$ of p-valent starlike functions are trivial. From it there also follows a simple and explicit example of a coefficient problem in $S(p)$ with no solution. This situation, which Goodman has treated in some detail [2], is interesting since such problems always have solutions in the schlicht case [3; 4].

Let S be the class of all functions $f(z)=z+a_2z^2+a_3z^3+\cdots$ which are regular and schlicht in $|z|<1$ and let S^* be the subclass of S consisting of those functions whose image domains are starshaped with respect to the origin. For a given positive integer p let $S(p)$, the class of p-valent starlike functions, be the class of all functions f to which there corresponds some r, $0<r<1$, such that for any z, $r<|z|<1$, $\operatorname{Re}\{zf'(z)/f(z)\}\geqq 0$ and $(1/2\pi)\int_0^{2\pi}\operatorname{Re}\{zf'(z)/f(z)\}dt=p, z=qe^{it}$, for each q, $r<q<1$. This integral is just the number of zeros of f in the interior of the circle $|z|=q$ and hence f has p zeros in the open unit disk, and is in fact p-valent there [2]. In a certain sense the classes $S(1)$ and S^* coincide, i.e., if $f\in S^*$ then $f\in S(1)$ and if $f\in S(1)$ then $f/f'(0)\in S^*$.

2. **The winding number.** If Q is a path in $U=\{z: |z|<1\}$ and f is analytic in U, let $f(Q)$ denote the path which is the image of Q under f and which has the induced orientation. The properties of the winding number

$$n[f(Q), a] = \frac{1}{2\pi i}\oint_Q \frac{f'(z)}{f(z)-a}\,dz$$

are well known. In this paper f will lack singularities and Q will be a circle. Hence $n[f(Q), a]$ will be the number of times f assumes the value a in the open disk bounded by Q.

If a function f is regular at a point $a\neq 0$ and $\operatorname{Re}\{zf'(z)/f(z)\}\geqq 0$ for all z in some neighborhood M of a then the fact that $\operatorname{Re}\{zf'(z)/f(z)\}$ is harmonic in some neighborhood $N\subset M$ of a implies that $\operatorname{Re}\{af'(a)/f(a)\}>0$ and, consequently, also that $f'(a)\neq 0$. From this fact, which will be of frequent use below, follows

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THEOREM 1. *Let the function f be regular in the open unit disk U and zero at the origin. Suppose there is an $r=r(f)$, $0<r<1$, such that* (i) $f(z)\neq 0$ *and* (ii) $\operatorname{Re}\{zf'(z)/f(z)\}\geqq 0$, *whenever $r<|z|<1$. Then for any q, $r<q<1$, and associated $Q=\{z: |z|=q\}$ and any $a\in U$, the winding number $n[f(Q), sf(a)]$ is a decreasing function of the positive real variable s as long as $sf(a)\in f(U)$.*

PROOF. Let $f(U)$ be the image of U under f. The winding number $n[f(Q), sf(a)]$ is a nonnegative integer (hence real) and is constant throughout each component of $f(U)$ determined by $f(q)$. Therefore, as has just been noted, $\operatorname{Re}\{zf'(z)/f(z)\}=\partial \arg f(z)/\partial \arg z>0$ whenever $r<|z|<1$, i.e. $\arg f$ is a strictly increasing function of $\arg z$ for $z\in Q$. Furthermore the fact that f is never zero in $\{z: r<|z|<1\}$ and has only a finite number $m>0$ of zeros in $\{z: |z|\leqq r\}$ implies, with the help of the argument principle, that $\arg f(z)$ increases by $2m\pi$ as z makes one positively directed circuit of Q. Thus $\operatorname{Arg} f(z)$ takes on each value b, $0\leqq b<2\pi$ precisely m times as z traverses Q.

If $a\in Q$ is arbitrary it is apparent that the angle θ from the radius vector $f(a)-0$ to the vector tangent to $f(Q)$ at $f(a)$ lies in the interval $0<\theta<\pi$. The geometric meaning of the winding number now makes it obvious that its value falls as $f(Q)$ is crossed in an outward direction and in fact that this decrease is just some integer n, $1\leqq n\leqq m$, which is the number of points of Q mapped into $f(a)$ by f. The proof of the theorem is now complete.

3. The class M_p^*.

DEFINITION 1. Let p be fixed, $p\geqq 1$. Suppose the function $h(z) =z^p+b_{p+1}z^{p+1}+b_{p+2}z^{p+2}+\cdots$ is regular in U and satisfies the conditions

(i) h is p-valent in U, and

(ii) there exists $r=r(h)$, $0<r<1$, such that $\operatorname{Re}\{zh'(z)/h(z)\}\geqq 0$ whenever $r<|z|<1$.

Then h is called a function of class M_p^*.

Manifestly if $h\in M_p^*$ then for any q, $r<q<1$, and $Q=\{z: |z|=q\}$ it must be that $n[h(Q), 0]=p$, i.e., $(1/2\pi)\int_0^{2\pi}\operatorname{Re}\{zh'(z)/h(z)\}dt=p$, $z=qe^{it}$, so that $h\in S(p)$. Thus $M_p^*\subset S(p)$. Let $(S^*)^p$ be the class of pth powers of functions of S^*. Then

THEOREM 2. $M_p^*=(S^*)^p, \qquad p=1, 2, 3, \cdots.$

PROOF. Choose any function

$$\begin{aligned} h(z) &= z^p + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots \\ &= z^p(1 + b_{p+1}z + b_{p+2}z^2 + \cdots) = z^p g(z) \end{aligned}$$

of M_p^*. Note that g is regular and nonzero in U. Hence so is $[g(z)]^{1/p}$. Since $g(0)=1$ the function $[g(z)]^{1/p}$ can be assumed to be 1 at the origin. Set $f(z)=z[g(z)]^{1/p}=z+\cdots$. Then $[f(z)]^p=h(z)$. Furthermore, if $z\in U$, $zh'(z)/h(z)=p(zf'(z)/f(z))$ whence, since $p>0$, $\operatorname{Re}\{zf'(z)/f(z)\}\geqq 0$ whenever $r(h)=r<|z|<1$. Hence, just as above, $\arg f(z)$ is a strictly increasing function of $\arg z$ for $z\in Q$. Therefore $f(Q)$ is a simple closed curve and by Darboux's Theorem f is schlicht. Thus $f\in S^*$, and consequently $M_p^*\subset(S^*)^p$.

If, now, $f(z)=z+a_2z^2+a_3z^3+\cdots$ is a member of S^* and the function $h(z)=z^p+\cdots$ is defined by setting $h(z)=[f(z)]^p$ then certainly $h\in M_p^*$, for f can take on each pth root of a given number at most once in U, showing that h satisfies condition (i). That h satisfies condition (ii) has already been shown in the first half of this proof. Therefore $(S^*)^p\subset M_p^*$ and the theorem is proved.

4. The class N_p^*.

Definition 2. Let p be fixed, $p\geqq 1$. Let m be an integer, $1\leqq m\leqq p$. Suppose the function $h(z)=z^m+b_{m+1}z^{m+1}+b_{m+2}z^{m+2}+\cdots$ is regular in U and satisfies the conditions

(i) h is at most p-valent in U, and

(ii) $\operatorname{Re}\{zh'(z)/h(z)\}\geqq 0$ for all $z\in U$.

Then h is called a function of class N_p^*.

The relationship of the class N_p^* to $S(p)$ is made plain by the following

Theorem 3. $N_p^*=S^*\cup(S^*)^2\cup\cdots\cup(S^*)^p, \qquad p=1,2,3,\cdots.$

Proof. Consider an arbitrary $h\in N_p^*$. The function h is of the form $h(z)=z^m+b_{m+1}z^{m+1}+b_{m+2}z^{m+2}+\cdots$ for some integer m, $1\leqq m\leqq p$. To show that $h\in M_m^*=(S^*)^m$ it clearly suffices to verify that h is m-valent in U. Consider any point $a\neq 0$ of U. If $h(a)=0$ then, in some neighborhood of a, $h(z)=(z-a)^ng(z)$ where $1\leqq n$ and $g(a)\neq 0$ (for the identically zero function is not a member of N_p^*). Then $zh'(z)/h(z)=na/(z-a)+n+zg'(z)/g(z)$. Certainly $n+zg'(z)/g(z)$ is analytic at $z=a$ since $g(a)\neq 0$ and therefore $zh'(z)/h(z)$ has a pole of order 1 at $z=a$ in contradiction to condition (ii) in the hypothesis concerning h. Thus h can be zero in U only at the origin. Hence for any q, $0<q<1$, and associated $Q=\{z\colon |z|=q\}$ and $G=\{z\colon |z|<q\}$ it is apparent that $n[h(Q),h(0)]=n[h(Q),0]=m$. Theorem 1 now guarantees that $n[h(Q),h(z)]\leqq m$ whenever $z\in G$. Since q can be arbitrarily close to 1, h must be m-valent in U. Thus $N_p^*\subset S^*\cup(S^*)^2\cup\cdots\cup(S^*)^p$.

For $m=1,2,3,\cdots,p$ the proof that $(S^*)^m\subset N_p^*$ is the same as the corresponding part of Theorem 2. This establishes the opposite inclusion and therewith the theorem.

5. The class S_p^*.

DEFINITION 3. Let $p=1, 2, 3, \cdots$. Suppose the function $f(z) = z+a_2z^2+a_3z^3+ \cdots$ is regular in U and satisfies the conditions

(i) f is at most p-valent in U,

(ii) there exists some $a \in f(U)$ such that $f(z)=a$ exactly p times in U, and

(iii) there exists $r=r(f)$, $0<r<1$, such that $\text{Re}\{zf'(z)/f(z)\} \geqq 0$ whenever $r<|z|<1$.

Then f is called a function of class S_p^*.

Evidently $S_1^*=S^*$ which, in turn, is in the sense above alluded to just equal to $S(1)$. But not even in this sense is $S_p^*=S(p)$. However

THEOREM 4. $S_p^* \subset S(p)$, $\qquad p=1, 2, 3, \cdots$.

PROOF. If $f \in S_p^*$ then all that must be verified to show that $f \in S(p)$ is the existence of a d, $0<d<1$, such that whenever $d<q<1$

$$n[f(Q), 0] = \frac{1}{2\pi}\int_0^{2\pi} \text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} dt = p,$$

$z = qe^{it}$.

There are some $a \in U$, some number k, $0<k<1$, and a circle $K=\{z: |z|=k\}$ contained in U such that $n[f(K), f(a)]=p$. Let c, $0<c<1$, be the largest of the moduli of the (at most p) points z_j of U at which $f(z_j)=0$. The number $r=r(f)$ is already associated with f. Define $d=\max\{r, k, c\}$. Then for any q, $d<q<1$, and associated $Q=\{z: |z|=q\}$ it is true that $n[f(Q), f(a)]=p$. The fact that f is at most p-valent in U implies that $n[f(Q), 0] \leqq p$. Hence an application of Theorem 1 to f yields $p=n[f(Q), f(a)] \leqq n[f(Q), 0] \leqq p$. Since q can be arbitrarily close to 1 this shows that $f \in S(p)$, i.e. $S_p \subset S(p)$, which completes the proof.

If g is any function of $S(p)$ having a simple zero at the origin then g can be written $g(z)=a_1z+a_2z^2+a_3z^3+ \cdots$. It is now a trivial matter to verify that

$$\frac{g(z)}{a_1} = \frac{g(z)}{g'(0)} = z + \frac{a_2}{a_1}z^2 + \frac{a_3}{a_1}z^3 + \cdots$$

is an element of S_p^*. And in this sense, the same as with the classes $S(1)$ and $S^*=S_1^*$ (i.e. except for normalization) the class S_p^* is just the class of all functions of $S(p)$ having a simple zero at the origin.

It is clear that for $p=2, 3, 4, \cdots$ no function of S_p^* is the power of a (schlicht) starlike function.

The following theorems give an example of a coefficient problem which has no solution.

THEOREM 5. *Let p be fixed, $p \geqq 2$. If k is any complex number such that $2-1/p<|k|$ then the function $f(z)=z+kz^p$ is a member of S_p^*.*

PROOF. The function f has p zeros in U and is obviously p-valent in U. Also, for $|z|=1$,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{p + \frac{(1-p)z}{z+kz^p}\right\} \geqq p - \frac{p-1}{|k|-1}$$
$$= \frac{p}{k-1}\left(|k| - 2 + \frac{1}{p}\right) > 0,$$

which persists, by continuity, for $r<|z|<1$, for some r, $0<r<1$. Therefore $f \in S_p^*$.

THEOREM 6. *Let p be fixed, $p \geqq 2$. Let $n \neq p$ be chosen, $n \geqq 2$. Then to any complex number q there corresponds a function $f(z)=z+a_2z^2+a_3z^3+\cdots$ of S_p^* for which $a_n=q$.*

PROOF. If $q=0$ the theorem is an immediate consequence of Theorem 5. So consider arbitrary fixed nonzero q, and let b be any complex number whose modulus satisfies the inequality $1+|q|<p+|q|+|n-p|\cdot|q|<|b|$. Now consider the function $f(z)=z+bz^p+qz^n$. In consequence of the above inequality and a theorem of Pellet [1, p. 10] on roots of polynomials it follows immediately that $n[f(C), 0]=p$, where $C=\{z: |z|=1\}$. The fact that $\max_{z\in C}|zf'(z)/f(z)-p|<1<p$ implies, just as in the proof of Theorem 5, that there exists d, $0<d<1$, such that $\operatorname{Re}\{zf'(z)/f(z)\} \geqq 0$ whenever $d<|z|<1$. Since f takes on only p zeros in U there exists t, $0<t<1$, such that $f(z)\neq 0$ whenever $t<|z|<1$. Set $r=\max(d, t)$. Then $0<r<1$ and Theorem 1 is applicable to f. Hence if $r<x<1$ and $X=\{z: |z|=x\}$ the winding number $n[f(X), sf(a)]$ is a decreasing function of the positive real variable s whenever $sf(a) \in f(U)$. Thus $p=n[f(C), 0]=n[f(X), 0] \geqq n[f(X), k]$ for any $k \in f(U)$. But x can be arbitrarily close to 1. Therefore $f \in S_p^*$ and the theorem is proved.

Consideration of the classes of p-valent starlike functions treated above has given rise to the following question concerning a decomposition for elements of $S(p)$. Given any $f \in S(p)$, does f have a representation $f=gh$ where $g \in S_p^*$, $h \in (S^*)^{p-1}$?

References

1. M. J. Dieudonné, *La théorie analytique des polynomes d'une variable*, Gauthier-Villars, Paris, 1938.

2. A. W. Goodman, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. **68** (1950), 204–233.

3. J. A. Hummel, *The coefficient regions of starlike functions*, Pacific J. Math. **7** (1957), 1381–1389.

4. ———, *A variational method for starlike functions*, Proc. Amer. Math. Soc. **9** (1958), 82–87.

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