FIXED POINT THEOREMS FOR PSEUDO MONOTONE MAPPINGS[1]

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1. **Introduction.** Recently [7] the author generalized a well-known theorem of Hamilton [1] in the following manner: *if X is a continuum each of whose subcontinua is unicoherent and decomposable, then X has the fixed point property for monotone transformations.* As a corollary it followed that the same fixed point property obtains for continua each of whose nondegenerate subcontinua has a cutpoint. The argument depended on the order structure of a certain arcwise connected hyperspace of the continuum.

In this note we arrive at the same corollary by a distinctly different and simpler proof. *Au fond* the argument is essentially the same as one due to Kelley [2] where it was shown that a homeomorphism of a continuum into itself has an invariant, cutpoint-free subcontinuum. (The analogous result for monotone transformations was proved by the author in [6].) The proof of Kelley does not make full use of the properties of homeomorphisms; the essential properties which make his argument work define a class of transformations which we shall term the *pseudo monotone* mappings.

Finally, we note that our results for pseudo monotone mappings admit a further generalization in the setting of partially ordered topological spaces.

2. **Pseudo monotone mappings.** Let X and Y be spaces and $f: X\rightarrow Y$ a continuous mapping. We say that f is *pseudo monotone* if, whenever A and B are closed and connected subsets of X and Y, respectively, and $B\subset f(A)$, it follows that some component of $A\cap f^{-1}(B)$ is mapped by f onto B. In general this notion is independent of that of a monotone mapping, but in certain applications of interest every monotone mapping is pseudo monotone.

Recall that a continuum (=compact connected Hausdorff space) is *hereditarily unicoherent* if any two of its subcontinua meet in a connected set.

LEMMA 1. *If X is an hereditarily unicoherent continuum and $f: X\rightarrow Y$ is a monotone mapping, then f is pseudo monotone.*

Presented to the Society, April 22, 1961; received by the editors December 28, 1960.

[1] This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF 49(638)-889. Reproduction in whole or in part is permitted for any purpose of the United States Government.

PROOF. Let A and B be closed and connected subsets of X and Y, respectively, such that $B \subset f(A)$. Since f is monotone, $f^{-1}(B)$ is a continuum, and since X is hereditarily unicoherent, $A \cap f^{-1}(B)$ is connected. Hence f is pseudo monotone.

Suppose now that X is a continuum and that $f: X \to X$ is continuous. A simple maximality argument establishes the existence of a nonempty subcontinuum Y, which is minimal with respect to being invariant under f. Suppose Y has a cutpoint p, with

$$Y - p = A \cup B$$

where A and B are disjoint, separated and nonempty. If $f(p) = p$ then the minimality of Y is contradicted, so we may assume $f(p) \in A$ and define $r(Y) = \overline{A}$ by

$$r(x) = x, \qquad x \in \overline{A},$$

$$r(x) = p, \qquad x \in \overline{B}.$$

The mapping $g: \overline{A} \to \overline{A}$ defined by $g = rf$ is continuous, and the set

$$K = \bigcap_{n=1}^{\infty} \{g^n(\overline{A})\}$$

is a subcontinuum of $\overline{A}$ which is invariant under g. Thus

$$f(K) \cap K = rf(K) = g(K) = K$$

and we infer $K \subset f(K)$. Therefore, *if f is pseudo monotone*, the set $K \cap f^{-1}(K)$ has a component K_1 such that $f(K_1) = K$. Inductively we obtain a sequence of subcontinua, K_n, such that

$$K_n \subset f(K_n) = K_{n-1} \subset \cdots \subset f(K_1) = K.$$

Clearly, the intersection of this sequence is a nonempty subcontinuum invariant under f, and this contradicts the minimality of Y. We have proved

THEOREM 1. *If X is a continuum and $f: X \to X$ is a pseudo monotone mapping, then X contains a nonempty subcontinuum Y which is minimal with respect to being invariant under f. Moreover, Y has no cutpoints.*

COROLLARY 1.1. *If X is a continuum such that each of its nondegenerate subcontinua has a cutpoint, and if $f: X \to X$ is a pseudo monotone mapping, then there exists $x_0 \in X$ such that $x_0 = f(x_0)$.*

It has been proved elsewhere [7] that the continua of Corollary 1.1 are hereditarily unicoherent. Therefore, by Lemma 1, we have

COROLLARY 1.2. *If X is a continuum such that each of its nondegenerate subcontinua has a cutpoint, and if $f: X \to X$ is a monotone mapping, then there exists $x_0 \in X$ such that $x_0 = f(x_0)$.*

3. **A generalization.** In [5] the author defined a POTS (= partially ordered topological space) to be a partially ordered set X, so topologized that the sets

$$L(x) = \{a: a \leqq x\}, \qquad M(x) = \{a: x \leqq a\}$$

are closed, for each $x \in X$. Two elements x and y of X are *comparable* if $x \leqq y$ or $y \leqq x$. In the event X contains a *unit*, i.e., a unique element e such that $L(e) = X$, we say that the subset A of X is *bounded away from* e if there exists $y \neq e$ such that $A \subset L(y)$.

The following theorem was proved in [5].

FIXED POINT THEOREM. *Let X be a compact Hausdorff* POTS *with unit, e. Let $f: X \to X$ be a continuous, order-preserving mapping satisfying the following conditions.*

(i) *There exists $x \neq e$ such that x and $f(x)$ are comparable.*

(ii) *If $x \neq e$ and if x and $f(x)$ are comparable, then either the sequence $f^n(x)$, $n = 1, 2, \cdots$, is bounded away from e, or $f^{-1}(x) \cap L(x)$ is nonempty.*

Then there exists $x_0 \neq e$ such that $x_0 = f(x_0)$.

For the remainder of this paper let us assume that X is a compact Hausdorff POTS with unit e, which is endowed with the following two properties.

(α) *There exist elements a, b and p of X such that $L(a) \cap L(b) = p$.*

(β) *If $x \in X - L(a) \cup L(b)$ then $p \leqq x$ and each of the sets $L(x) \cap L(a)$ and $L(x) \cap L(b)$ has a supremum.*

Let $f: X \to X$ be continuous and order-preserving, and suppose f maps minimal elements into minimal elements. In addition, suppose f satisfies the following order-theoretic analogue of pseudo monotonicity.

(P) *If $x \leqq f(x)$ then $f^{-1}(x) \cap L(x)$ is nonempty.*

According to the fixed point theorem above, f has a fixed point distinct from e if $f(x) \leqq x$ for some $x \neq e$. If this does not occur, then by (β) and the fact that $f(p)$ is minimal, we have $f(p) \leqq a$ or $f(p) \leqq b$, but not both. Suppose $f(p) \leqq a$; since f is order-preserving, $f(a)$ cannot lie in $L(b)$. Moreover, $f(a)$ cannot lie in $L(a)$ by assumption, so that by (β) there must exist

$$t_1 = \sup (L(f(a)) \cap L(a)),$$

with $p \leqq t_1$. Now $f(t_1) \in X - L(a)$ and, since $p \leqq t_1$, it follows that $f(p) \leqq f(t_1)$ and hence $f(t_1) \in X - L(b)$. Applying (β) again there exists

$$t_2 = \sup(L(f(t_1)) \cap L(a))$$

with $p \leqq t_2$. Because f is order-preserving it follows that $f(t_1) \leqq f(a)$ and hence $t_2 \leqq f(a)$. Moreover, $t_2 \leqq a$ so that $t_2 \leqq t_1$. Inductively, we obtain a sequence t_n satisfying

$$t_{n+1} = \sup(L(f(t_n)) \cap L(a)), p \leqq t_{n+1} \leqq t_n.$$

Since t_n is a decreasing sequence, it must converge to some $t_0 \leqq t_n$. Further, since $t_n \leqq f(t_{n-1})$, it follows that $t_0 \leqq f(t_0)$. Condition (i) is now satisfied and (ii) follows from (P) and the above discussion. Hence we infer (compare with a result of A. D. Wallace [**4**])

THEOREM 2. *Let X be a nondegenerate compact Hausdorff* POTS *with unit e, satisfying* (α) *and* (β). *Let $f: X \rightarrow X$ be a continuous, order-preserving mapping which maps minimal elements into minimal elements and satisfies* (P). *Then there exists $x_0 \in X - e$ such that $f(x_0) = x_0$.*

It is not difficult to see that Theorem 2 is truly a generalization of Theorem 1. Let Y be a continuum with a cutpoint p, and let $f(Y) = Y$ be pseudo monotone. If X is the space of subcontinua of Y, endowed with the finite topology [**3**], and if f^* is the mapping of X into itself induced by f, then X and f^* satisfy the hypotheses of Theorem 2, where the partial order is taken to be inclusion. Thus Y contains an invariant proper subcontinuum and Theorem 1 follows.

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