REMARKS ON A PAPER OF HOBBY AND WRIGHT

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C. Hobby and C. R. B. Wright [2] have just published the following Theorem A. However, their proof seems to contain an error.[2]

The notation of [2] is used except that G_n is not reserved for the nth term of the lower central series of G: $\phi(G)$ denotes the Frattini subgroup of G; (G, H) means the group generated by the commutators $g^{-1}h^{-1}gh$ where $g \in G$, $h \in H$; $(A_1, A_2, \cdots, A_{n+1})$ is defined inductively as $((A_1, A_2, \cdots, A_n), A_{n+1})$; $H \subset G$ means that H is properly included in G.

THEOREM A. *If G is a finite p-group and H a subgroup of G such that $H_n \subset G_n$, then $(H\phi(G))_n \subset G_n$, where X_n denotes the nth term of the lower central series of X.*

N. Itô [3] had already proved this theorem for the case $n=2$. In this note, Itô's theorem is generalized in a somewhat stronger form than Theorem A. In fact, as was shown in [2], if Theorem A were false, it would have to fail for a normal subgroup H of G. In the presence of this fact, Theorem A is contained in

THEOREM B. *Let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$ be a nondecreasing finite chain of normal subgroups of a finite p-group G and let $H_1, H_2, \cdots, H_n$ be normal subgroups of G with $H_i \subseteq G_i$ for all i. If*

$$(H_1, H_2, \cdots, H_n) \subset (G_1, G_2, \cdots, G_n),$$

then

$$(H_1\phi(G_1), H_2\phi(G_2), \cdots, H_n\phi(G_n)) \subset (G_1, G_2, \cdots, G_n).$$

PROOF. Suppose that the theorem is false for a certain n. Let G be of minimal order for which it is false and let $H_1, H_2, \cdots, H_n$ be chosen such that if K_i is a normal subgroup of G and $H_i \subset K_i \subseteq G_i$ for any i, then

$$(H_1, H_2, \cdots, K_i, \cdots, H_n) = (G_1, G_2, \cdots, G_n).$$

For convenience, set $(H_1, H_2, \cdots, H_n) = A$, $(H_1\phi(G_1), H_2\phi(G_2), \cdots, H_n\phi(G_n)) = B$, and $(G_1, G_2, \cdots, G_n) = C$. First, it is noted that

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[2] The fact that G is noncyclic does not imply that $\phi(G)$ is the intersection of all normal subgroups of index p^2 in G.

$$(H_1/A, \cdots, H_n/A) = A/A \subset C/A = (G_1/A, \cdots, G_n/A).$$

Thus if $A \neq \langle 1 \rangle$, the relation

$$B/A = (H_1\phi(G_1)/A, \cdots, H_n\phi(G_n)/A) \subset C/A$$

holds according to the choice of G since in general $\phi(G/N) = N\phi(G)/N$. However, this implies that $B \subset C$; hence $A = \langle 1 \rangle$.

Let z be an element of order p in Z, the center of G. If the relation

$$\begin{aligned}\langle z \rangle/\langle z \rangle &= (\langle z \rangle H_1/\langle z \rangle, \cdots, \langle z \rangle H_n/\langle z \rangle) \\ &\subset (\langle z \rangle G_1/\langle z \rangle, \cdots, \langle z \rangle G_{n-1}/\langle z \rangle, G_n/\langle z \rangle) = \langle z \rangle C/\langle z \rangle\end{aligned}$$

holds, then

$$\langle z \rangle B/\langle z \rangle = (\langle z \rangle H_1\phi(G_1)/\langle z \rangle, \cdots, \langle z \rangle H_n\phi(G_n)/\langle z \rangle) \subset \langle z \rangle C/\langle z \rangle,$$

which implies that $B \subset C$. Thus $C = \langle z \rangle$.

Since all the subgroups involved are normal, from the linearity properties of commutators [1, p. 150] it follows that

$$\begin{aligned}&(H_1, \cdots, H_{k-1}, H_k\phi(G_k), H_{k+1}, \cdots, H_n) \\ &\qquad = A(H_1, \cdots, H_{k-1}, \phi(G_k), H_{k+1}, \cdots, H_n) = \langle 1 \rangle.\end{aligned}$$

But $H_k \subset H_k\phi(G_k)$ for some k. Thus a contradiction has been established on the choice of the H_i, and the theorem is proved.

References

1. M. Hall, *The theory of groups*, New York, Macmillan, 1959.

2. C. Hobby and C. R. B. Wright, *A generalization of a theorem of N. Itô on p-groups*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 707–709.

3. N. Itô, *On a theorem of L. Rédei and J. Szép concerning p-groups*, Acta Sci. Math. Szeged. vol. 14 (1952) pp. 186–187.

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