TESTS FOR THE SUPERADDITIVITY OF FUNCTIONS

A. M. BRUCKNER

1. **Introduction.** A function f defined on an interval $I \equiv [0, a]$ is called superadditive if $f(x+y) \geqq f(x)+f(y)$ whenever x, y and $x+y$ are in I. A simple example of a superadditive function is furnished by a convex function f with $f(0) \leqq 0$. More generally, a function star-shaped with respect to the origin is superadditive. Superadditive functions have been studied by Hille and Phillips [2][1] and Rosenbaum [3], but tests for superadditivity are not given in those studies. In this paper we derive conditions for the superadditivity of a function. Our method is to decompose a function f into several component functions in a certain manner and to give a condition which states that the superadditivity of each component function along with the satisfaction of a side condition guarantees the superadditivity of f. We then examine questions concerned with the superadditivity of convexo-concave functions. The results show that the condition is relatively easy to apply whenever the component functions can be chosen to be convexo-concave. No other nontrivial sufficient conditions for the superadditivity of a function are known to the author.

2. **Minimal superadditive extensions of superadditive functions.** In what follows we will make use of the notion of the minimal superadditive extension of a superadditive function. This notion has been studied by the author [1]. We will summarize those results of the study which will be needed.

Let f be superadditive on $[0, a]$. Then there exists a function F with the following properties:

(a) $F \equiv f$ on $[0, a]$,

(b) F is superadditive on $[0, \infty)$,

(c) If g is a function satisfying the conditions $g \equiv f$ on $[0, a]$ and g is superadditive on $[0, \infty)$ then $F \leqq g$ on $[0, \infty)$.

F is called the minimal superadditive extension of f.

We will make use of the following

THEOREM. *Let f be a continuous non-negative superadditive function on $[0, a]$ and let F be its minimal superadditive extension. For each $x \in [0, \infty)$, there exists a finite number of points $x_1, x_2, \cdots, x_N$ such that $x = x_1+x_2+\cdots+x_N$, $0 \leqq x_i \leqq a$ for $i=1, 2, \cdots, N$ and $F(x) = f(x_1)+f(x_2)+\cdots+f(x_N)$. If f is differentiable at two of these points, say x_j and x_k, then $f'(x_j) = f'(x_k)$.*

Received by the editors February 1, 1961.

[1] Numbers in brackets refer to the bibliography at the end.

The points $x_1, x_2, \cdots, x_N$ are said to form a decomposition of x. The distinct nonzero points of the set $\{x_1, x_2, \cdots, x_N\}$ are called the members of the decomposition.

3. Decompositions of functions.

DEFINITION. Let f be defined on $[0, a]$. The functions $f_1, f_2, \cdots, f_p$ defined on $[0, a_1]$, $[0, a_2]$, $\cdots$, $[0, a_p]$ respectively form a decomposition of the function f provided $a_1+a_2+\cdots+a_p=a$ and

$$f(x) = \begin{cases} f_1(x), \\ f_2(x-a_1)+f_1(a_1), \quad 0 \leqq x \leqq a_1,\ a_1 < x \leqq a_1+a_2, \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ f_p(x-a_1-a_2-\cdots-a_{p-1})+f_1(a_1)+\cdots+f_{p-1}(a_{p-1}), \\ \qquad\qquad a_1+\cdots+a_{p-1} < x \leqq a. \end{cases}$$

If $f_1, f_2, \cdots, f_p$ form a decomposition for f, we write $f=f_1\wedge f_2\wedge \cdots \wedge f_p$. The functions $f_1, f_2, \cdots, f_p$ are called the component functions of the decomposition. Geometrically the graph of f is the graph obtained by joining the graphs of $f_1, f_2, \cdots, f_p$ end-to-end provided $f_k(0)=0$ and f_k continuous $k=1, 2, \cdots, p$.

4. A condition for the superadditivity of a function. We now combine the notions described in the preceding two sections.

THEOREM 1. *Let f_1 and f_2 be non-negative superadditive functions defined on $[0, a_1]$ and $[0, a_2]$ respectively and let $f=f_1\wedge f_2$. Denote by F_1 the minimal superadditive extension of f_1. A necessary and sufficient condition that f be superadditive on $[0, a_1+a_2]$ is that $f \geqq F_1$ on $[0, a_1+a_2]$.*

PROOF. The necessity of the condition is obvious. Let x, y and $x+y$ be in the interval $[0, a_1+a_2]$, with say, $x\leqq y$. We wish to show $f(x+y)\geqq f(x)+f(y)$. If $x+y\leqq a_1$, then the validity of this inequality follows from the superadditivity of f_1. So we turn to the case $a_1<x+y$. If $y<a_1$ we have

$$f(x+y)-f(y) \geqq F_1(x+y)-F_1(y) \geqq F_1(x) = f(x).$$

The first inequality follows from the hypothesis and the second from the superadditivity of F_1. Next, if $y\geqq a_1\geqq x$, then we have

$$f(x+y)-f(y) \geqq f(a_1+x)-f(a_1) \geqq F_1(a_1+x)-F_1(a_1) \geqq f(x).$$

Here the first inequality follows from the superadditivity of f_2, the second from the hypotheses, and the third from the superadditivity of F_1. Finally, if $a_1 \leqq x$, then

$$\begin{aligned} f(x+y) - f(y) &= f(x+y) - f(a_1+y) + f(a_1+y) - f(y) \\ &\geqq f(x) - f(a_1) + f(2a_1) - f(a_1) \\ &= f(x) + f(2a_1) - 2f(a_1) \geqq f(x), \end{aligned}$$

the inequalities following from the superadditivity of f_2.

This completes the proof of Theorem 1.

The following theorem can be obtained from Theorem 1 by an induction argument.

THEOREM 2. *Let $f_1, f_2, \cdots, f_p$ be non-negative and superadditive on $[0, a_1], [0, a_2], \cdots, [0, a_p]$ respectively and let $f = f_1 \wedge f_2 \wedge \cdots \wedge f_p$. Denote by F_K the minimal superadditive extension of f_K, $K = 1, 2, \cdots, p$. Then f is superadditive on $(0, a_1 + a_2 + \cdots + a_p)$ provided $f_K \wedge \cdots \wedge f_p \geqq F_K$ for each $K = 1, 2, \cdots, p$.*

5. **Convexo-concave functions.** The theorems of the preceding section are useful only if the components satisfy two requirements: first, that their superadditivity can be readily checked; and, second, that their minimal superadditive extensions can be readily obtained. In this section we see that a convexo-concave function satisfies both requirements: the superadditivity of a convexo-concave function can be ascertained by checking only some combinations of x and y in the inequality defining superadditivity, and the minimal superadditive extension of a superadditive convexo-concave function can be calculated using decompositions having at most two members. The results of this section apply whenever the function whose superadditivity we wish to establish is decomposable into convexo-concave component functions.

DEFINITION. A continuous function f defined on $[0, a]$ is called convexo-concave if there exists a number b, $0 \leqq b \leqq a$ such that f is convex on $[0, b]$ and concave on $[b, a]$.

THEOREM 3. *Let f be a convexo-concave function defined on the interval $I = [0, a]$ with $f(0) \leqq 0$. Then a necessary and sufficient condition that f be superadditive is that* $\max_{x \in I} [f(x) + f(a-x)] \leqq f(a)$.

PROOF. The necessity of the condition is obvious. To prove the sufficiency of the condition, consider the function g defined on the set

$$T = \{(x, y) : 0 \leqq x, y, x + y \leqq a\}$$

by

$$g(x, y) \equiv f(x+y) - f(x) - f(y).$$

Our condition states $g \geqq 0$ on the set $\{(x, y): x+y=a\}$. Also, $g(0, 0) \geqq 0$ since $f(0) \leqq 0$. Now, it is easy to check that g is either increasing, or decreasing, or increasing and then decreasing in each of the variables, holding the other variable fixed. Hence, for fixed x, g attains its minimum at $(x, 0)$ or $(x, a-x)$ and a similar statement holds for fixed y. It follows that the minimum value of g on T is attained at a point on the line $x+y=a$, or at the origin. Thus, $g \geqq 0$ on T and f is superadditive on $[0, a]$.

Again, let f be convexo-concave on $[0, a]$, $f(0) \leqq 0$. Let n be any positive integer such that f is superadditive on the set

$$V = \left\{0, \frac{a}{n}, \frac{2a}{n}, \cdots, \frac{n-1}{n} a, a\right\} \qquad (n = 1 \text{ will always work}).$$

Define a set S in the plane as follows: For each pair j, k of integers, $0 \leqq j, k, j+k \leqq n-2$ we consider the square which is the convex hull of the points

$$\left(\frac{j}{n} a, \frac{k}{n} a\right), \quad \left(\frac{j+1}{n} a, \frac{k}{n} a\right), \quad \left(\frac{j}{n} a, \frac{k+1}{n} a\right)$$

and

$$\left(\frac{j+1}{n} a, \frac{k+1}{n} a\right).$$

Let S be the union of these squares. The points determining the squares comprising S will be called corners of S. By using the monotonicity behavior in each variable of the function g defined in Theorem 3, we note that on each square of S, g attains its minimum at a corner of the square. Now, since f is superadditive on V, $g \geqq 0$ on the corners of S, hence $g \geqq 0$ on S. But this implies f is superadditive on the interval $[0, (n-1)a/n]$. We have proved

THEOREM 4. *If the convexo-concave function f defined on $[0, a]$ is superadditive on the discrete set of points $\{0, a/n, 2a/n, \cdots, (n-1)a/n, a\}$ and $f(0) \leqq 0$, then f is superadditive on the interval $[0, (n-1)a/n]$.*

We can combine Theorems 3 and 4.

THEOREM 5. *Let f be convexo-concave on $[0, a]$ with $f(0)=0$. If $f(ka/n)+f((n-k)a/n) \leqq f(a)$ for some positive integer n and all $k=0, 1, \cdots, n$, then f is superadditive on the interval $[0, (n-1)a/n]$.*

PROOF. Let p be a polygonal function whose "vertices" are $(ka/n, f(ka/n))$ for $k=0, 1, \cdots, n$. The function p is convexo-concave on $[0, a]$. Now, by hypothesis, $p(a-x)+p(x) \leqq p(a)$ whenever $x=ka/n$, $k=0, 1, \cdots, n$. This implies $p(a-x)+p(x) \leqq p(a)$ for all $x \in [0, a]$. For a proof of this statement the reader is referred to the proof of Theorem 8, [1]. By Theorem 3, p is superadditive on $[0, a]$ so that f is superadditive on $[0, (n-1)a/n]$ by Theorem 4.

In particular, it is clear from the continuity of f at a, that if the hypotheses of Theorem 5 are satisfied by every positive integer n, then f is superadditive on $[0, a]$.

The minimal superadditive extension of a differentiable strictly convexo-concave function is easy to compute. Using the theorem of §2, it is easy to show that if $z > a$, a decomposition for z can contain at most two members. If there are two, one must be the end point a, and the other in the interval of convexity of f. In fact, if the inflection point is at $x = u < a/2$, then a decomposition for z must consist of a single member.

BIBLIOGRAPHY

1. Andrew Bruckner, *Minimal superadditive extensions of superadditive functions*, Pacific J. Math. **10** (1960), 1155–1162.

2. E. Hille and R. Phillips, *Functional analysis and semigroups*, Chapter VII, pp. 237–255, Amer. Math. Soc. Colloq. Publ., Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

3. R. A. Rosenbaum, *Subadditive functions*, Duke Math. J. **17** (1950), 227–247.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA