A PROPERTY OF HOMOGENEOUS PROCESSES

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1. Statement of results. In the following G is a locally compact Hausdorff group, K a compact subgroup, $X=G/K$ the homogeneous space of left cosets, and $C(X)$ the Banach space of continuous complex valued functions on X which are constant at infinity. $(P_t)_{t\geq 0}$ denotes a homogeneous process on X. That is, P_t: $C(X)\to C(X)$ is a strongly continuous one parameter semi-group of positive, constant-preserving linear transformations of $C(X)$ which commute with left translation by elements of G. Stated in other words, $(P_t)_{t\geq 0}$ is a strongly continuous one parameter semi-group on $C(X)$; $f\geqq 0$ implies $P_tf\geqq 0$; $P_t1=1$ and $L_gP_t=P_tL_g$ where $L_gf\cdot(x)=f(g^{-1}[x])$, $f\in C(X)$, $g\in G$, $x\in X$.

P_t is represented by a kernel $P_t(x, A)$,

$$P_tf\cdot(x) = \int P_t(x, dz)f(z),$$

which is the transition probability of a stationary Markov process on X. For the kernel, homogeneity means $P_t(x, A)=P_t(g[x], g[A])$ when $x\in X$, $g\in G$ and A is a Borel subset of X. It is shown below that

THEOREM. *Every homogeneous process possesses Property* II.

PROPERTY II. *For each $z\in X$ there is a regular Borel measure Q_z on $X-\{z\}$ such that*

$$t^{-1}P_tf\cdot(z) \to Q_z(f) \qquad \textit{as } t\to 0, \tag{1.1}$$

for each $f\in C(X)$ which vanishes on a neighborhood of z. Q_z is not necessarily bounded but it is bounded on the complement of any neighborhood of z.

The stochastic and analytic implications of Property II are discussed in [1]. Roughly speaking, Q_x describes very precisely the nature of the discontinuities in the paths of any process with transition probabilities $P_t(x, A)$, while from the analytic point of view Q_x is related to the form of the infinitesimal generator of P_t, and Property II implies, for example, that the domain of this infinitesimal generator admits very satisfying smoothing operations.

2. Reduction and reformulation. By way of preliminary computations, let H be a compact subgroup of G and dh the normalized Haar

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measure of H. Associated with H there are two projection operators on $C(G)$. Namely, $f \to R_H f$ and $f \to L_H f$. These are defined by

$$R_H f\cdot(g) = \int R_h f\cdot(g)dh, \qquad R_h f\cdot(g) = f(gh).$$

$$L_H f\cdot(g) = \int L_h f\cdot(g)dh, \qquad L_h f\cdot(g) = f(h^{-1}g).$$

When H is an invariant subgroup the automorphism $h \to ghg^{-1}$, $g \in G$, preserves the normalized Haar measure, so that $\int f(gh)dh = \int f(hg)dh$ and $L_H = R_H$. Similar computations show that when H is invariant $R_{HK} = L_H R_K$. A function $f \in C(G)$ is constant on the left cosets of G modulo K if and only if $R_K f = f$. Thus the two spaces $R_K C(G)$ and $C(X)$ are isomorphic, and using this isomorphism there is a one-to-one correspondence between homogeneous processes on X and positive, constant-preserving semi-groups $(P_t)_{t \geq 0}$ on $C(G)$ which satisfy $L_g P_t = P_t L_g$, $g \in G$; $R_K P_t = P_t R_K = P_t$; and which are strongly continuous on $C(G)$ when $t > 0$ and on the subspace $R_K C(G)$ at $t = 0$. We call the latter a K-homogeneous process on G.

Because of the homogeneity it suffices to prove (1.1) for a fixed $z \in X$, say for the coset K of G/K. Furthermore, it suffices to prove the limit (1.1) exists and is finite. The positivity of P_t can then be used to show this limit has the form $Q_z(f)$ described in the statement of Property II. With these modifications the theorem can be restated.

RESTATEMENT OF THE THEOREM. *If P_t is a K-homogeneous process on G and $f \in R_K C(G)$ vanishes on a neighborhood of K, then*

$$t^{-1} P_t f\cdot(e) \to \textit{a finite limit} \qquad \textit{as } t \to 0. \tag{2.1}$$

One further reduction is necessary before proceeding. This is to notice that it suffices to prove the restatement of the theorem when G is σ-compact (a countable union of compact sets). To see this let A_t be a σ-compact set in G on which the measure $P_t(e, \cdot)$ is concentrated, and let G' be the subgroup of G generated by some compact neighborhood D of K and the A_t, t-rational. G' is closed and σ-compact and contains the support of every $P_t(e, \cdot)$, $t \geq 0$, because $P_t f\cdot(e)$ is continuous in $t > 0$. By homogeneity the support of the functional $f \to P_t f\cdot(g)$, $g \in G'$, is also contained in G' and, in fact, $P_t(g, A)$, $g \in G'$, A Borel in G', defines a K-homogeneous process on G' with a unique extension to G. Clearly the limit (2.1) is unaffected by the values of f outside G' and one may as well assume G is σ-compact.

3. **Proof of the theorem.** This theorem has already been proved when X is separable in [1, §3]; so it is sufficient here to reduce the proof to the separable case. The key element in this proof is the following lemma which was suggested to the author by an argument in [2, p. 58].

LEMMA. *Let G be a locally compact, Hausdorff, σ-compact topological group and let $f \in C(G)$. Then there is a compact invariant subgroup N of G such that $L_N f = f$ and G/N is separable.*

To prove (2.1) from the lemma, simply note that $P_t' = P_t L_N = L_N P_t$ defines an NK-homogeneous process on G and that $f = L_N f = L_N R_K f = R_{NK} f$ vanishes on a neighborhood of NK. Since G/NK is separable, the separable version of (2.1) as proved in [1] implies that

$$t^{-1} P_t L_N f \cdot (e) = t^{-1} P_t f \cdot (e) \to \text{a limit} \qquad \text{as } t \to 0.$$

4. **Proof of the lemma.** f is constant at infinity and hence uniformly continuous on G. Let W_n be a compact neighborhood of the identity e such that for every $g \in G$, $|f(gk) - f(g)| < 1/n$ when $k \in W_n$. We shall prove there is a compact invariant subgroup $N \subset \bigcap_n W_n$ such that G/N is separable. Clearly for any such N, $L_N f = f$. To show the existence of N let C_n be an increasing sequence of compact sets which cover G and choose V_n inductively so that

(1) V_n is a compact symmetric neighborhood of e.
(2) $V_n^2 \subset V_{n-1} \cap W_n$.
(3) $g^{-1} V_n g \subset V_{n-1}$ for every $g \in C_n$.

$N = \bigcap_n V_n$ is a compact invariant subgroup of G. If T is the canonical projection $G \to G/N$, the sets $T(V_n)$ form a basis for the neighborhoods at the identity in G/N because for each open $U \supset N$, $V_n N - U$ is a decreasing sequence of compact sets with empty intersection and consequently $V_n N \subset U$ for some n. Since G/N is a σ-compact uniform space with a countable basis for its uniformity it follows that G/N is separable. Alternatively, G/N is a σ-compact metrizable space and hence separable.

REFERENCES

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